MODULE-4

Introduction

Let V and W be vector spaces and let T be a function from V to W. The V is called the domain of T and W the codomain of T.

Let $v \in V$ and $w \in W$ & $T(v) = w$.

Then w is called the image of v under T.

The set of all images of vectors in V is called the range of T.

For a given $w \in W$ the set of all v in V such that $T(v) = w$ is called the preimage of w.

Problems

1. Let $V = (v_1, v_2) \in \mathbb{R}^2$. If $T: V \rightarrow W$ is defined as

$T(v_1, v_2) = (v_1 + v_2, 2v_2)$

1. Find the image of $v = (1, -3)$
2. Find the image of $v = (0, 0)$
3. Find the pre image of $w = (1, 3)$.

Ans: $T(v_1, v_2) = (v_1 + v_2, 2v_2)$

1). $T(1, -3) = (1 + -3, 2 \times -3) = (-2, -6)$

2) $T(0, 0) = (0 + 0, 2 \times 0) = (0, 0)$

3) To find the pre image of $w = (1, 3)$ consider set of all (v_1, v_2) such that $T(v_1, v_2) = (1, 3)$

$$\Rightarrow (v_1 + v_2, 2v_2) = (1, 3)$$

$$\Rightarrow v_1 + v_2 = 1, \quad 2v_2 = 3$$

$\Rightarrow V_2 = \frac{3}{2}$

$V_1 + V_2 = 1 \Rightarrow V_1 + \frac{3}{2} = 1$

$\Rightarrow V_1 = 1 - \frac{3}{2} = \frac{-1}{2}$

$\therefore$ Preimage of $w = (1, 3)$ is $\underline{\underline{\left(\frac{-1}{2}, \frac{3}{2}\right)}}$

hw

2. Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. If $T: V \to W$ is defined as

$T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$

a) Find the image of $v = (-4, 5, 1)$

b) Find the pre image of $w = (4, 1, -1)$

Ans: $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$

a) $T(-4, 5, 1) = (2 \times -4 + 5, 2 \times 5 - 3 \times -4, -4 - 1)$

$= \underline{\underline{(-3, 22, -5)}}$

b) To find pre image of $w = (4, 1, -1)$ consider set of all (v_1, v_2, v_3) $\ni$ $T(v_1, v_2, v_3) = (4, 1, -1)$

$\Rightarrow (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3) = (4, 1, -1)$

$\Rightarrow 2v_1 + v_2 = 4$

$2v_2 - 3v_1 = 1 \Rightarrow -3v_1 + 2v_2 = 1$

$v_1 - v_3 = -1$

$\therefore AX = B \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ -3 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

Augmented matrix $=$ ~~2/~~ $\begin{bmatrix} 2 & 1 & 0 & 4 \\ -3 & 2 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}$ $R_1 \to \frac{R_1}{2}$

$$\sim \begin{bmatrix} 1 & 1/2 & 0 & 2 \\ -3 & 2 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 0 & 2 \\ 0 & 7/2 & 0 & 7 \\ 0 & -1/2 & -1 & -3 \end{bmatrix} \quad R_2 \to R_2 \times \frac{2}{7}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & -1/2 & -1 & -3 \end{bmatrix} \quad R_3 \to R_3 + \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad R_3 \to \frac{R_3}{-1}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore\ V_1 + \frac{1}{2} V_2 = 2$

$V_2 = 2$

$V_3 = 2$

$\therefore\ V_1 + \frac{1}{2} \cdot 2 = 2 \Rightarrow V_1 = 2 - 1 = 1$

$\therefore$ Pre image of $\omega = (4, 1, -1)$ is $\underline{(1, 2, 2)}$

Linear Transformation

Let V and W be vector spaces. The function $T: V \to W$ is a linear transformation of V into W when the following two properties holds for all $u, v \in V$ and for any scalar c

1) $T(u+v) = T(u) + T(v)$

2) $T(cu) = cT(u)$.

Linear Operator

A linear transformation $T: V \to V$ from a vector space V onto itself is called a linear operator.

Problems

1. Show that the function $T(v_1, v_2) = (v_1 + v_2, v_1 - 2v_2)$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$.

Ans: ~~It~~ Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}^2$ and let c be any real number.

$$T(u+v) = T(u_1 + v_1, u_2 + v_2)$$

$$= (u_1 + v_1 + u_2 + v_2, u_1 + v_1 - 2(u_2 + v_2))$$

$$= (u_1 + u_2 + v_1 + v_2, u_1 - 2u_2 + v_1 - 2v_2)$$

$$= (u_1 + u_2, u_1 - 2u_2) + (v_1 + v_2, v_1 - 2v_2)$$

$$= T(u_1, u_2) + T(v_1, v_2)$$

$$= T(u) + T(v)$$

$$T(cu) = T(c(u_1, u_2))$$

$$= T(cu_1, cu_2)$$

$$= (cu_1 + cu_2, cu_1 - 2cu_2)$$

$$= c(u_1 + u_2, u_1 - 2u_2)$$

$$= c\,T(u_1, u_2) = c\,T(u)$$

$\therefore T$ is a linear transformation.

HW

2. Determine whether the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, 1)$ is a linear transformation.

Ans: Let $u = (u_1, u_2)$ and $v = (v_1, v_2) \in \mathbb{R}^2$.

$T(u+v) = T((u_1, u_2) + (v_1, v_2))$

$= T(u_1+v_1, u_2+v_2)$

$= (u_1+v_1, 1)$

$T(u) + T(v) = T(u_1, u_2) + T(v_1, v_2)$

$= (u_1, 1) + (v_1, 1)$

$= (u_1+v_1, 2)$

$\therefore T(u+v) \neq T(u) + T(v)$.

So T is not a linear transformation.

3. Determine whether the function $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x^2, xy, y^2)$ is a linear transformation.

Ans: Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}^2$

$T(u+v) = T((u_1, u_2) + (v_1, v_2))$

$= T(u_1+v_1, u_2+v_2)$

$= ((u_1+v_1)^2, (u_1+v_1)(u_2+v_2), (u_2+v_2)^2)$

$T(u) + T(v) = T(u_1, u_2) + T(v_1, v_2)$

$= (u_1^2, u_1u_2, u_2^2) + (v_1^2, v_1v_2, v_2^2)$

$= (u_1^2+v_1^2, u_1u_2+v_1v_2, u_2^2+v_2^2)$

$\therefore T(u+v) \neq T(u) + T(v)$

So T is not a linear transformation.

HW

4. Check whether $T: M_{2,2} \to \mathbb{R}$ defined by $T(A) = a+b+c+d$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a linear transformation.

Ans: Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{2,2}$

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right)$$

$$= a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + d_1 + d_2$$

$$T(A_1) + T(A_2) = T\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + T\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$= a_1 + b_1 + c_1 + d_1 + a_2 + b_2 + c_2 + d_2$$

$$\Rightarrow T(A_1 + A_2) = T(A_1) + T(A_2)$$

$$T(cA_1) = T\left[c\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right]$$

$$= T\begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} = ca_1 + cb_1 + cc_1 + cd_1$$

$$cT(A_1) = cT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= c(a_1 + b_1 + c_1 + d_1) = ca_1 + cb_1 + cc_1 + cd_1$$

$\therefore T(cA_1) = cT(A_1)$

$\therefore T$ is a linear transformation

5. Let $u \in \mathbb{R}^2$ and $v = (1,1)$. Check whether $T(u) = \text{Proj}_v u$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Ans: $\text{Proj}_v u = \dfrac{\langle u, v \rangle}{\langle v, v \rangle} v$

$$T(u_1+u_2) = \operatorname{Proj}_v (u_1+u_2)$$

$$= \frac{\langle u_1+u_2, v\rangle}{\langle v, v\rangle} v$$

$$= \left(\frac{\langle u_1, v\rangle}{\langle v, v\rangle} + \frac{\langle u_2, v\rangle}{\langle v, v\rangle}\right) v$$

$$= \frac{\langle u_1, v\rangle}{\langle v, v\rangle} v + \frac{\langle u_2, v\rangle}{\langle v, v\rangle} v$$

$$= \operatorname{Proj}_v u_1 + \operatorname{Proj}_v u_2$$

$$= T(u_1) + T(u_2)$$

$$T(cu) = \operatorname{Proj}_v cu$$

$$= \frac{\langle cu, v\rangle}{\langle v, v\rangle} v$$

$$= c\frac{\langle u, v\rangle}{\langle v, v\rangle} v$$

$$= c \operatorname{Proj}_v u$$

$$= cT(u)$$

$\therefore$ T is a linear transformation.

<u>Note</u>

Let V and W be vector spaces and let T be a linear transformation from V into W, and let $u, v \in V$. Then 1) $T(0) = 0$

2) $T(-v) = -T(v)$

3) $T(u-v) = T(u) - T(v)$

Problems

HW 1. Let $T: R^3 \to R^3$ be a linear transformation such that $T(1,0,0) = (3,1,-4)$, $T(0,1,0) = (-1,3,-2)$, $T(0,0,1) = (0,2,1)$. Find $T(1,2,-3)$.

Ans: Expressing the vector $(1,2,-3)$ as a linear combination of the vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ we have

$$(1,2,-3) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

$$= (c_1, c_2, c_3)$$

$$\Rightarrow c_1 = 1, \quad c_2 = 2, \quad c_3 = -3$$

$$\therefore (1,2,-3) = 1(1,0,0) + 2(0,1,0) - 3(0,0,1)$$

$$\therefore T(1,2,-3) = 1T(1,0,0) + 2T(0,1,0) - 3T(0,0,1)$$

$$= 1(3,1,-4) + 2(-1,3,-2) - 3(0,2,1)$$

$$= (3,1,-4) + (-2,6,-4) - (0,6,3)$$

$$= \underline{\underline{(1,1,-11)}}$$

2. Let $T: R^3 \to R^3$ be a linear transformation such that $T(1,1,1) = (2,0,-1)$, $T(0,-1,2) = (-3,2,-1)$, $T(1,0,1) = (1,1,0)$. Find $T(4,2,0)$.

Ans: Expressing the vector $(4,2,0)$ as a linear combination of the vectors $(1,1,1)$, $(0,-1,2)$ and $(1,0,1)$ we have

$$(4,2,0) = c_1(1,1,1) + c_2(0,-1,2) + c_3(1,0,1)$$

$= (c_1, c_1, c_1) + (0, -c_2, 2c_2) + (c_3, 0, c_3)$

$= (c_1 + c_3, c_1 - c_2, c_1 + 2c_2 + c_3)$

$\Rightarrow c_1 + c_3 = 4$

$c_1 - c_2 = 2$

$c_1 + 2c_2 + c_3 = 0$

Using Gauss elimination method, the augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ $\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & -1 & -1 & -2 \\ 0 & 2 & 0 & -4 \end{bmatrix}$

$R_2 \to \frac{R_2}{-1}$ $\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & -4 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -8 \end{bmatrix}$

$R_3 \to \frac{R_3}{-2}$ $\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$\therefore c_1 + c_3 = 4$

$c_2 + c_3 = 2$

$c_3 = 4$

$\therefore c_2 + 4 = 2 \Rightarrow c_2 = 2 - 4 = -2$

$c_1 + 4 = 4 \Rightarrow c_1 = 4 - 4 = 0$

$\therefore (4, 2, 0) = 0(1, 1, 1) - 2(0, -1, 2) + 4(1, 0, 1)$

$\therefore T(4, 2, 0) = 0T(1, 1, 1) - 2T(0, -1, 2) + 4T(1, 0, 1)$

$\Rightarrow T(4,2,0) = 0 - 2(-3,2,-1) + 4(1,1,0)$

$= (6,-4,2) + (4,4,0)$

$= \underline{\underline{(10,0,2)}}$

HW 3. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1,0) = (1,1)$, $T(0,1) = (-1,1)$. Find $T(x,y)$ and hence evaluate $T(1,4)$ and $T(-2,1)$.

Ans: Expressing $(x,y) \in \mathbb{R}^2$ as a linear combination of $(1,0)$ and $(0,1)$, we have

$(x,y) = c_1(1,0) + c_2(0,1)$

$= (c_1, 0) + (0, c_2)$

$= (c_1, c_2)$

$\Rightarrow c_1 = x, \quad c_2 = y$

$\therefore (x,y) = x(1,0) + y(0,1)$

$\Rightarrow T(x,y) = xT(1,0) + yT(0,1)$

$= x(1,1) + y(-1,+1)$

$= (x,x) + (-y,y)$

$= (x-y, x+y)$

$\therefore T(1,4) = (1-4, 1+4) = (-3,5)$

$T(-2,1) = (-2-1, -2+1)$

$= \underline{\underline{(-3,-1)}}$

H/W

4. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1,2) = (1,0)$, $T(-1,1) = (0,1)$. Find $T(x,y)$ and hence evaluate $T(2,0)$.

Ans: Expressing (x,y) as a linear combination of $(1,2)$ and $(-1,1)$ we have

$$(x,y) = c_1(1,2) + c_2(-1,1)$$
$$= (c_1, 2c_1) + (-c_2, c_2)$$
$$= (c_1 - c_2, 2c_1 + c_2)$$

$$\Rightarrow c_1 - c_2 = x$$
$$2c_1 + c_2 = y$$

~~Adding these~~

Using Gauss elimination method, augmented matrix is $\begin{bmatrix} 1 & -1 & x \\ 2 & 1 & y \end{bmatrix}$

$R_2 \to R_2 - 2R_1 \sim \begin{bmatrix} 1 & -1 & x \\ 0 & 3 & y-2x \end{bmatrix}$

$R_2 \to \frac{R_2}{3} \sim \begin{bmatrix} 1 & -1 & x \\ 0 & 1 & \frac{y-2x}{3} \end{bmatrix}$

$$\Rightarrow c_1 - c_2 = x$$
$$c_2 = \frac{1}{3}(y-2x)$$

$$\therefore c_1 - \frac{1}{3}(y-2x) = x$$
$$\Rightarrow c_1 = x + \frac{1}{3}y - \frac{2}{3}x$$
$$= \frac{1}{3}x + \frac{1}{3}y = \frac{1}{3}(x+y)$$

$\therefore (x,y) = \frac{1}{3}(x+y)\,(1,2) + \frac{1}{3}(y-2x)\,(-1,1)$

$\Rightarrow T(x,y) = \frac{1}{3}(x+y)\,T(1,2) + \frac{1}{3}(y-2x)\,T(-1,1)$

$= \frac{1}{3}(x+y)\,(1,0) + \frac{1}{3}(y-2x)\,(0,1)$

$= \left(\frac{x+y}{3}, 0\right) + \left(0, \frac{y-2x}{3}\right)$

$= \left(\frac{x+y}{3}, \frac{y-2x}{3}\right)$

$\therefore T(2,0) = \left(\frac{2+0}{3}, \frac{0-4}{3}\right) = \underline{\underline{\left(\frac{2}{3}, \frac{-4}{3}\right)}}$

HW

5\. Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1,0,0) = (1,0,-1)$, $T(0,1,0) = (2,1,1)$, $T(0,0,1) = (1,-1,0)$. Hence find $T(2,-2,1)$.

Ans: Expressing $(x,y,z) \in \mathbb{R}^3$ as a linear combination of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ we have

$(x,y,z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$= (c_1, 0, 0) + (0, c_2, 0) + (0, 0, c_3)$

$= (c_1, c_2, c_3)$

$\therefore c_1 = x, \quad c_2 = y, \quad c_3 = z$

$\therefore (x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$$\Rightarrow T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(1,0,-1) + y(2,1,1) + z(1,-1,0)$$
$$= (x,0,-x) + (2y,y,y) + (z,-z,0)$$
$$= (x+2y+z,\ y-z,\ -x+y)$$

$$T(2,-2,1) = (2-4+1,\ -2-1,\ -2-2)$$
$$= \underline{\underline{(-1,-3,-4)}}$$

6. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(1,2) = (1,0,1)$, $T(-1,1) = (0,1,-1)$. Find $T(1,1)$ and $T(0,0)$.

Ans: Expressing $(x,y) \in \mathbb{R}^2$ as a linear combination of $(1,2)$ and $(-1,1)$ we have

$$(x,y) = c_1(1,2) + c_2(-1,1)$$
$$= (c_1, 2c_1) + (-c_2, c_2)$$
$$= (c_1 - c_2,\ 2c_1 + c_2)$$

$$\Rightarrow c_1 - c_2 = x$$
$$2c_1 + c_2 = y$$

By Gauss elimination method, augmented matrix is

$$\begin{bmatrix} 1 & -1 & x \\ 2 & 1 & y \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \sim \begin{bmatrix} 1 & -1 & x \\ 0 & 3 & y-2x \end{bmatrix}$$

$R_2 \to \frac{R_2}{3}$ $\sim \begin{bmatrix} 1 & -1 & x \\ 0 & 1 & \frac{y-2x}{3} \end{bmatrix}$

$\Rightarrow c_1 - c_2 = x$

$c_2 = \frac{y-2x}{3}$

$\therefore c_1 - \left(\frac{y-2x}{3}\right) = x$

$\Rightarrow c_1 = x + \frac{y-2x}{3} = x + \frac{y}{3} - \frac{2x}{3}$

$= \frac{y}{3} + \frac{x}{3} = \frac{x+y}{3}$

$\therefore (x,y) = \left(\frac{x+y}{3}\right)(1,2) + \left(\frac{y-2x}{3}\right)(-1,1)$

$= \cancel{\left(\frac{x+y}{3}, \frac{2(x+y)}{3}\right)} + \cancel{\left(\frac{-(y-2x)}{3}, \frac{y-2x}{3}\right)}$

$= \cancel{x+y-y}$

$\therefore T(x,y) = \left(\frac{x+y}{3}\right) T(1,2) + \left(\frac{y-2x}{3}\right) T(-1,1)$

$= \left(\frac{x+y}{3}\right)(1,0,1) + \left(\frac{y-2x}{3}\right)(0,1,-1)$

$= \left(\frac{x+y}{3}, 0, \frac{x+y}{3}\right) + \left(0, \frac{y-2x}{3}, -\frac{(y-2x)}{3}\right)$

$= \left(\frac{x+y}{3}, \frac{y-2x}{3}, \frac{x+y-y+2x}{3}\right)$

$= \left(\frac{x+y}{3}, \frac{y-2x}{3}, x\right)$

$\therefore T(1,1) = \left(\frac{2}{3}, \frac{-1}{3}, 1\right)$

$T(0,0) = (0,0,0)$

7. Find the linear transformation $T : \mathbb{R}^2 \to M_{2,2}$ such that $T(1,0) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $T(2,-1) = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$.

Hence find $T(1,1)$.

Ans: Expressing $(x,y) \in \mathbb{R}^2$ as a linear combination of $(1,0)$ and $(2,-1)$ we have

$$(x,y) = c_1(1,0) + c_2(2,-1)$$
$$= (c_1, 0) + (2c_2, -c_2)$$
$$= (c_1 + 2c_2, -c_2)$$

$$\Rightarrow c_1 + 2c_2 = x$$
$$-c_2 = y \Rightarrow c_2 = -y$$

$$\therefore c_1 - 2y = x \Rightarrow c_1 = x + 2y$$

$$\therefore (x,y) = (x+2y)(1,0) + -y(2,-1)$$

$$\therefore T(x,y) = (x+2y)\,T(1,0) - y\,T(2,-1)$$
$$= (x+2y)\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} - y\begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} x+2y & 2x+4y \\ -x-2y & 0 \end{bmatrix} - \begin{bmatrix} -y & 3y \\ 0 & 2y \end{bmatrix}$$
$$= \begin{bmatrix} x+3y & 2x+y \\ -x-2y & -2y \end{bmatrix}$$

$$\therefore T(1,1) = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

Linear Transformation given by a matrix

Let A be an $m \times n$ matrix. Then the function $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(v) = Av$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Problems

1. Define the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(v) = Av$ where $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$. Find the dimensions n and m of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Find $T(v)$ where $v \in \mathbb{R}^n$.

Ans: Since A is a 3×2 matrix, $m = 3$ and $n = 2$.

$\therefore$ Dimension of $\mathbb{R}^n$ is 2 and dimension of $\mathbb{R}^m$ is 3.

Let $v = (x, y) \in \mathbb{R}^2$

$$\therefore T(v) = Av = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ -2x + 4y \\ -2x + 2y \end{bmatrix}$$

$\therefore T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined as

$$T(x, y) = (x + 2y, -2x + 4y, -2x + 2y)$$

HW

2. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(v) = Av$ where $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Find (a) $T(1,1)$ (b) the pre-image of $(1, 2)$, (c) the pre-image of $(0,0)$.

Ans: $T(v) = Av$, where $v \in \mathbb{R}^2$.

Let $v = (x, y)$

$\therefore T(V) = AV = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$

$\Rightarrow T(x,y) = (-y, -x)$.

(a) $T(1,1) = (-1,-1)$

(b) Pre-image of $(1,2)$ is $\{(x,y) \in \mathbb{R}^2 \ \& \ T(x,y) = (1,2)\}$

ie; $(-y,-x) = (1,2)$

$\Rightarrow -y = 1, \ -x = 2$

$\Rightarrow y = -1, \ x = -2$

$\therefore$ Preimage of $(1,2) = \underline{\underline{(-2,-1)}}$

(c) Pre-image of $(0,0)$ is $\{(x,y) \in \mathbb{R}^2 \ \& \ T(x,y) = (0,0)\}$

ie; $(-y,-x) = (0,0)$

$\Rightarrow -y = 0, \ -x = 0$

$\Rightarrow y = 0, \ x = 0$

$\therefore$ Pre image of $(0,0) = \underline{\underline{(0,0)}}$

3 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(V) = AV$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. Find (a) $T(1,1,1)$ (b) the pre image of $(1,3)$ (c) the pre-image of $(0,0)$.

Ans. Given $T(V) = AV$ where $V \in \mathbb{R}^3$. Let $V = (x,y,z)$

$\therefore T(x,y,z) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \end{bmatrix}$

$\therefore T(x,y,z) = (x+y, y-z)$

(a) $T(1,1,1) = (1+1, 1-1) = (2,0)$

(b) Let the pre-image of $(1,3)$ be (x,y,z).

Then $T(x,y,z) = (1,3)$

$\Rightarrow (x+y,\ y-z) = (1,3)$

$\Rightarrow x+y=1\ ,\quad y-z=3$

$\Rightarrow x=1-y\ ,\quad z=y-3$

Let $y=k$, then $x=1-k$ and $z=k-3$

$\therefore$ The pre image of $(1,3) = \underline{\{(1-k,\ k,\ k-3)/k\in\mathbb{R}\}}$

(c) The pre image of $(0,0)$ be (x,y,z).

Then $T(x,y,z) = (0,0)$

$\Rightarrow (x+y,\ y-z) = (0,0)$

$\Rightarrow x+y=0,\ y-z=0$

$\Rightarrow x=-y,\quad z=y$

Let $y=k$, then $x=-k,\ z=k$

ie, The preimage of $(0,0) = \{(-k,k,k)/k\in\mathbb{R}\}$

4. For the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ given by the matrix $A=\begin{bmatrix} a & -b \\ b & a\end{bmatrix}$, find a and b Such that $T(12,5) = (13,0)$.

Ans: Given $T:\mathbb{R}^2\to\mathbb{R}^2$ and $A=\begin{bmatrix} a & -b \\ b & a\end{bmatrix}$

For $(x,y)\in\mathbb{R}^2$, $T(x,y) = \begin{bmatrix} a & -b \\ b & a\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}$

$= \begin{bmatrix} ax-by \\ bx+ay\end{bmatrix}$

$T(12,5) = (13,0) \Rightarrow \begin{bmatrix} a.12-b.5 \\ b.12+a.5\end{bmatrix} = \begin{bmatrix} 13 \\ 0\end{bmatrix}$

$\Rightarrow 12a - 5b = 13 \qquad \Rightarrow \qquad 12a - 5b = 13$

$12b + 5a = 0 \qquad\qquad\qquad 5a + 12b = 0$

Augmented matrix $= \begin{bmatrix} 12 & -5 & 13 \\ 5 & 12 & 0 \end{bmatrix}$ $R_1 \to \frac{R_1}{12}$

$\sim \begin{bmatrix} 1 & -5/12 & 13/12 \\ 5 & 12 & 0 \end{bmatrix}$ $R_2 \to R_2 - 5R_1$

$\sim \begin{bmatrix} 1 & -5/12 & 13/12 \\ 0 & 169/12 & -65/12 \end{bmatrix}$ $R_2 \to R_2 \cdot \frac{12}{169}$

$\sim \begin{bmatrix} 1 & -5/12 & 13/12 \\ 0 & 1 & -65/169 \end{bmatrix}$

$\Rightarrow a - \frac{5}{12} b = \frac{13}{12}$

$b = \frac{-65}{169} = \frac{-5}{13}$

$\therefore a = \frac{13}{12} + \frac{5}{12} b = \frac{13}{12} - \frac{25}{12 \times 13}$

$= \frac{12}{13}$

$\therefore a = \frac{12}{13}$ and $b = \frac{-5}{13}$

HW

5. For the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} a & 2 \\ 3 & a+b \end{bmatrix}$, find a and b such that $T(1,2) = (5,7)$.

Ans: Given $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $A = \begin{bmatrix} a & 2 \\ 3 & a+b \end{bmatrix}$

$\therefore$ For $(x,y) \in \mathbb{R}^2$, $T(x,y) = \begin{bmatrix} a & 2 \\ 3 & a+b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} ax + 2y \\ 3x + (a+b)y \end{bmatrix}$

$T(1,2) = (5,7) \Rightarrow \begin{bmatrix} a\cdot 1 + 2\cdot 2 \\ 3\cdot 1 + (a+b)2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$\Rightarrow a + 4 = 5$

$3 + (a+b)2 = 7$

$a+4 = 5 \Rightarrow a = 5-4 = 1$

$\therefore 3 + (1+b)2 = 7 \Rightarrow 3 + 2 + 2b = 7$

$\Rightarrow 5 + 2b = 7$

$\Rightarrow 2b = 7-5 = 2$

$\Rightarrow b = \frac{2}{2} = 1$

$\therefore a = 1,\ b = 1$

Rotation in $\mathbb{R}^2$

Standard rotation matrix for a counter clockwise rotation by angle θ is

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

1. Find the linear transformation which rotates every vector in $\mathbb{R}^2$ counter clockwise about the origin through the angle θ. Also find LT when $\theta = 45^\circ$

Ans: Linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined as,

for $(x,y) \in \mathbb{R}^2$

$$T(x,y) = R(\theta)\cdot\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$\therefore T(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

when $\theta = 45^\circ$

$T(x,y) = (x\cos 45^\circ - y\sin 45^\circ,\ x\sin 45^\circ + y\cos 45^\circ)$

$= \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}},\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)$

Projection in $\mathbb{R}^3$

The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ represented by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is defined by

$T(v) = Av$

ie; $T(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

$\therefore T(x,y,z) = (x,y,0)$

we can consider the point $(x,y,0)$ in $\mathbb{R}^3$ as the point (x,y) in the xy-plane. ie; T orthogonally project every vector in $\mathbb{R}^3$ to xy-plane $\mathbb{R}^2$.

Similarly if $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the projection is to the yz-plane and if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the projection is to the xz-plane.

Kernel and Nullity

Kernel of a linear transformation $T: V \to W$ is the set of all vectors $v \in V$ that are mapped by T into the zero vector in W and is denoted as $\ker(T)$.

$$\ker(T) = \{ v \in V : T(v) = 0 \}$$

Note

1) kernel of a linear transformation is always non-empty, since if T is a linear transformation $T(0) = 0$. $\therefore$ The zero vector belongs to the kernel.
2) kernel of LT $T: V \to W$ is a subspace of V.

Nullity

Let $T: V \to W$ be a linear transformation. The dimension of kernel of T is called nullity of T. If $\ker(T)$ contains only zero vector, then nullity of $T = 0$.

Problems

1. Find the kernel of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1 - 2x_2, 3x_1, -5x_1)$

Ans: To find $\ker(T)$ we need to find all $v = (x_1, x_2)$ in $\mathbb{R}^2$ such that $T(x_1, x_2) = 0$

$$\Rightarrow (x_1 - 2x_2, 3x_1, -5x_1) = (0, 0, 0)$$

$$\Rightarrow \quad x_1 - 2x_2 = 0$$
$$3x_1 = 0$$
$$-5x_1 = 0$$

$\therefore x_1 = 0.$ $\quad x_1 - 2x_2 = 0$

$$\Rightarrow 0 - 2x_2 = 0 \Rightarrow x_2 = 0$$

$$\therefore \ker(T) = \{(0,0)\} = \{0\}$$

2. Find the kernel and nullity of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(X) = AX$ where $A = \begin{bmatrix} 3 & -3 & -6 \\ -1 & 2 & 3 \end{bmatrix}$.

Ans: To find $\ker(T)$, we need to find all $X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that $T(x_1, x_2, x_3) = 0$

$$\Rightarrow \begin{bmatrix} 3 & -3 & -6 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_1 - 3x_2 - 6x_3 \\ -x_1 + 2x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 - 3x_2 - 6x_3 = 0$$
$$-x_1 + 2x_2 + 3x_3 = 0$$

By Gauss elimination method, augmented matrix

$$= \begin{bmatrix} 3 & -3 & -6 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{3}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad x_1 - x_2 - 2x_3 = 0$$
$$x_2 + x_3 = 0$$

Put $x_2 = t \quad \therefore \quad x_2 + x_3 = 0$

$$\Rightarrow x_3 = -x_2 = -t$$

Also $x_1 - x_2 - 2x_3 = 0 \Rightarrow x_1 - t + 2t = 0$

$$\Rightarrow x_1 + t = 0 \Rightarrow x_1 = -t$$

$\therefore \ker T = \{(-t, t, t) : t \in \mathbb{R}\}$

$= \{t(-1, 1, 1) : t \in \mathbb{R}\} = \text{span}\{(1, -1, 1)\}$

So kernel of T is a subspace of $\mathbb{R}^3$ spanned by a vector $(1,-1,1)$.

$\therefore$ Dimension of kernel of $T = 1$

i.e; Nullity of $T = 1$

3 Find the kernel and nullity of the linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^4$ defined by $T(x) = Ax$ where $x \in \mathbb{R}^5$ and

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 & -1 \\ -3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Ans: To find ker(T), we need to find $x = (x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ such that $T(x_1, x_2, x_3, x_4, x_5) = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 & -1 \\ -3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + x_2 + 3x_3 + x_4 \\ x_1 + 2x_2 + x_4 - x_5 \\ -3x_1 - 6x_3 + 3x_5 \\ 2x_4 + 8x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 + 3x_3 + x_4 = 0$$
$$x_1 + 2x_2 + x_4 - x_5 = 0$$
$$3x_1 - 6x_3 + 3x_5 = 0$$
$$2x_4 + 8x_5 = 0$$

using Gauss elimination method, augmented matrix is

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & -1 & 0 \\ -3 & 0 & -6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 1 & 2 & 0 & 1 & -1 & 0 \\ -3 & 0 & -6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & -3/2 & 1/2 & -1 & 0 \\ 0 & 3/2 & -3/2 & 3/2 & 3 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \quad R_2 \to R_2 \times \frac{2}{3}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 1/3 & -2/3 & 0 \\ 0 & 3/2 & -3/2 & 3/2 & 3 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{3}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \quad R_4 \to R_4 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank $= 3 <$ no. of unknowns $= 5$. So we have to assign arbitrary values to $5 - 3 = 2$ variables.

$$x_1 + \frac{1}{2} x_2 + \frac{3}{2} x_3 + \frac{1}{2} x_4 = 0$$

$$x_2 - x_3 + \frac{1}{3} x_4 - \frac{2}{3} x_5 = 0$$

$$x_4 + 4x_5 = 0$$

Let $x_3 = s$ and $x_5 = t$

$\therefore x_4 + 4x_5 = 0 \Rightarrow x_4 = -4x_5 = -4t$

$x_2 - x_3 + \frac{1}{3}x_4 - \frac{2}{3}x_5 = 0$

$\Rightarrow x_2 - s + \frac{1}{3}\times -4t - \frac{2}{3}\times t = 0$

$\Rightarrow x_2 - s - \frac{4t}{3} - \frac{2t}{3} = 0$

$\Rightarrow x_2 - s - \frac{6t}{3} = 0$

$\Rightarrow x_2 = s + 2t$

$x_1 + \frac{1}{2}x_2 + \frac{3}{2}x_3 + \frac{1}{2}x_4 = 0$

$\Rightarrow x_1 + \frac{1}{2}(s+2t) + \frac{3}{2}(s) + \frac{1}{2}(-4t) = 0$

$\Rightarrow x_1 + \frac{1}{2}s + t + \frac{3}{2}s - 2t = 0$

$\Rightarrow x_1 + 2s - t = 0$

$\Rightarrow x_1 = -2s + t$

$\therefore \ker T = \{(-2s+t, s+2t, s, -4t, t) \mid s, t \in \mathbb{R}\}$

ie; $X = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

$\therefore B = \{(-2,1,1,0,0), (1,2,0,-4,1)\}$ is a basis for the kernel of T

$\therefore$ Nullity of $T = 2$

4. Find a basis for the kernel and nullity of the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(X) = AX$, where $X \in \mathbb{R}^4$ and $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 2 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 5 & 0 & -2 & 2 \end{bmatrix}$

Ans: To find ker(T), we need to find $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $T(x_1, x_2, x_3, x_4) = 0$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 2 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 5 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 + x_4 = 0$$

$$3x_1 + 2x_2 + x_3 - x_4 = 0$$

$$x_2 - x_3 + 2x_4 = 0$$

$$5x_1 - 2x_3 + 2x_4 = 0$$

Using Gauss elimination method, augmented matrix

$$= \begin{bmatrix} 2 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 5 & 0 & -2 & 2 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 3 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 5 & 0 & -2 & 2 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 7/2 & 1 & -5/2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 5/2 & -2 & -1/2 & 0 \end{bmatrix} \quad R_2 \to R_2 \times \frac{2}{7}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 2/7 & -5/7 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 5/2 & -2 & -1/2 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - \frac{5}{2} R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 2/7 & -5/7 & 0 \\ 0 & 0 & -9/7 & 19/7 & 0 \\ 0 & 0 & -19/7 & 9/7 & 0 \end{bmatrix} \quad R_3 \to R_3 \times \frac{-7}{9}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 2/7 & -5/7 & 0 \\ 0 & 0 & 1 & -19/9 & 0 \\ 0 & 0 & -19/7 & 9/7 & 0 \end{bmatrix} \quad R_4 \to R_4 + \frac{19}{7} R_3$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 2/7 & -5/7 & 0 \\ 0 & 0 & 1 & -19/9 & 0 \\ 0 & 0 & 0 & -35/19 & 0 \end{bmatrix} \quad R_4 \to R_4 \times \frac{-19}{35}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 2/7 & -5/7 & 0 \\ 0 & 0 & 1 & -19/9 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ Rank = no. of unknowns.

$$x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 = 0$$

$$x_2 + \frac{2}{7}x_3 - \frac{5}{7}x_4 = 0$$

$$x_3 - \frac{19}{9}x_4 = 0$$

$$x_4 = 0$$

$$x_3 - \frac{19}{9}x_4 = 0 \Rightarrow x_3 - 0 = 0 \Rightarrow x_3 = 0$$

$$x_2 + \frac{2}{7}x_3 - \frac{5}{7}x_4 = 0 \Rightarrow x_2 + 0 - 0 = 0 \Rightarrow x_2 = 0$$

$$x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 = 0 \Rightarrow x_1 - 0 + 0 = 0 \Rightarrow x_1 = 0$$

$\therefore \ker T = \{(0,0,0,0)\}$

So Nullity of $T = 0$

Range of a linear transformation.

The range of a linear transformation $T: V \to W$ is the set of vectors in W which are images of elements of V. ie; $range(T) = \{T(v) : v \in V\}$

Rank of a linear transformation.

Let $T: V \to W$ be a linear transformation. The dimension of range of T is called rank of T, denoted by $r(T)$.

Note

1. The range of a linear transformation $T: V \to W$ is a subspace of W.
2. If $T: V \to W$ is a linear transformation defined by $T(v) = Av$, then the range of T is equal to the subspace spanned by the column vectors of A. (ie; the column space of A).
3. Let A be an $m \times n$ matrix. Then the columns of A corresponding to the columns of leading variables of its row echelon form constitute a basis for the column space of A.

Steps to find the basis for the range of a LT

1) Write the given linear transformation in the form $T(v) = Av$
2. Reduce the matrix A to its row echelon form R.
3. Note the columns of R corresponding to the

leading variables.

4. Form the basis for the range of T using the corresponding columns of A.

Sum of Rank and Nullity (Rank Nullity Theorem)

Let $T: V \Rightarrow W$ be a linear transformation from an n-dimensional vector space V into a vector space W. Then the sum of the dimensions of the range of T and kernel of T is equal to the dimension of the domain V.

ie; $\text{Rank}(T) + \text{Nullity}(T) = n$.

Problems

1. Find the range and rank of the linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^4$ defined by $T(x) = Ax$, where $x \in \mathbb{R}^5$ and

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 & -1 \\ -3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Ans: Given $T(x) = Ax$ where $A = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 & -1 \\ -3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$

Now we have to reduce A to its echelon form.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 & -1 \\ -3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \quad R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 \\ 1 & 2 & 0 & 1 & -1 \\ -3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 \\ 0 & 3/2 & -3/2 & 1/2 & -1 \\ 0 & 3/2 & -3/2 & 3/2 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} R_2 \to R_2 \times \frac{2}{3}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/3 & -2/3 \\ 0 & 3/2 & -3/2 & 3/2 & 3 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} R_3 \to R_3 - \frac{3}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} R_4 \to R_4 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the leading variables in the row echelon form corresponds to the first, second and fourth columns of A.

$\therefore$ Basis of range of $T = \{(2,1,-3,0), (1,2,0,0), (1,1,0,2)\}$

$\therefore$ Rank of $T = \underline{\underline{3}}$

HW 2. Find a basis for the range and rank of the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(X) = AX$, where $X \in \mathbb{R}^4$ and $A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & 1 & -3 & -1 \\ -2 & 0 & 3 & -2 \end{bmatrix}$

Ans: Reducing A to its echelon form,

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & 1 & -3 & -1 \\ -2 & 0 & 3 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & -3 & -4 \\ 0 & -2 & 3 & 4 \end{bmatrix} R_2 \to \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -3/2 & -2 \\ 0 & -2 & 3 & 4 \end{bmatrix} R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -3/2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the leading variables in the row echelon form corresponds to the first and second columns of A.

$\therefore$ Basis for range of $T = \{(1, 1, -2), (-1, 1, 0)\}$

$\therefore$ Rank of $T = \underline{\underline{2}}$

3. Let $T: \mathbb{R}^5 \to \mathbb{R}^8$ be a linear transformation.

(a) Find the dimension of the kernel of T when the dimension of the range is 3.

(b) Find the rank of T when the nullity of T is 2

(c) Find the rank of T when $\ker(T) = \{0\}$.

Ans: Since $T: \mathbb{R}^5 \to \mathbb{R}^8$, dimension of domain = 5

ie; $n = 5$.

We have $\text{Rank}(T) + \text{Nullity}(T) = 5$ —— ①

a) Given dimension of the range = 3

ie; $\text{Rank}(T) = 3$

$\therefore$ ① $\Rightarrow 3 + \text{Nullity}(T) = 5$

$\Rightarrow \text{Nullity}(T) = 5 - 3 = 2$.

ie; dimension of $\ker(T) = \underline{\underline{2}}$

b) Given $\text{Nullity}(T) = 2$

$\therefore$ ① $\Rightarrow \text{Rank}(T) + 2 = 5$

$\Rightarrow \text{Rank}(T) = 5 - 2 = \underline{\underline{3}}$

(c) $\ker(T) = \{0\} \Rightarrow$ dimension of $\ker(T) = 0$

ie; $\text{Nullity}(T) = 0$

$\therefore$ ① $\Rightarrow \text{Rank}(T) + 0 = 5$

$\Rightarrow \text{Rank}(T) = \underline{\underline{5}}$

4. The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$

where $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find (a) $\ker T$ (b) $\text{Nullity}(T)$

(c) range (T) (d) rank (T). Also verify that rank (T) + Nullity (T) = dim ($\mathbb{R}^2$).

Ans: a) To find ker(T), we need to find $(x_1, x_2) \in \mathbb{R}^2$ such that $T(x_1, x_2) = 0$

$$\Rightarrow \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5x_1 - 3x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 3x_2 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 - x_2 = 0$$

Augmented matrix $= \begin{bmatrix} 5 & -3 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ $R_1 \to \frac{R_1}{5}$

$\sim \begin{bmatrix} 1 & -3/5 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & -3/5 & 0 \\ 0 & 8/5 & 0 \\ 0 & -2/5 & 0 \end{bmatrix}$ $R_2 \to R_2 \times \frac{5}{8}$

$\sim \begin{bmatrix} 1 & -3/5 & 0 \\ 0 & 1 & 0 \\ 0 & -2/5 & 0 \end{bmatrix}$ $R_3 \to R_3 + \frac{2}{5} R_2$

$\sim \begin{bmatrix} 1 & -3/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x_1 - \frac{3}{5} x_2 = 0$

$x_2 = 0$

$\therefore x_1 - 0 = 0 \Rightarrow x_1 = 0$

$\therefore \ker(T) = \{(0,0)\}$

(b) Nullity $(T) = 0$

(c) From part (a) $\begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore$ leading variables in row echelon form corresponds to the first and second columns of A

$\therefore$ Basis for range $(T) = \{(5,1,1), (-3,1,-1)\}$

(d). Rank $(T) = 2$

Now rank (T) + Nullity $(T) = 2 + 0 = 2 = \dim(\mathbb{R}^2)$.

Standard Matrix of a linear Transformation

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $B = \{e_1, e_2, \dots e_n\}$ be the standard basis for $\mathbb{R}^n$. Then standard matrix for a linear transformation is defined by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

where

$$T(e_i) = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} \text{ for } i = 1, 2, \dots n.$$

$T(V) = AV$ for all $V \in \mathbb{R}^n$.

Problems

1. Find the standard matrix for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x - 2y, 3x + y - z)$.

Ans: ~~Let~~ The standard basis of $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$

$\therefore e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$T(e_1) = T(1,0,0) = (1-0, 3+0-0) = (1,3)$

$T(e_2) = T(0,1,0) = (-2, 1)$

$T(e_3) = T(0,0,1) = (0, -1)$

$\therefore$ standard matrix $A = [T(e_1)\ T(e_2)\ T(e_3)]$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$$

2. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y) = (x-y, x+3y)$. Find the standard matrix for T.

Ans: The standard basis of $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$

$\therefore e_1 = (1,0)$, $e_2 = (0,1)$

$T(e_1) = T(1,0) = (1,1)$

$T(e_2) = T(0,1) = (-1,3)$

$\therefore$ standard matrix $A = [T(e_1) \ T(e_2)]$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

3. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y) = (x+y, 2x-y)$. Find the standard matrix for T.

Ans: The standard basis of $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$

$\therefore e_1 = (1,0)$, $e_2 = (0,1)$

$T(e_1) = T(1,0) = (1,2)$

$T(e_2) = T(0,1) = (1,-1)$

$\therefore$ standard matrix $A = [T(e_1) \ T(e_2)]$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

4. Use the standard matrix for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (2x+y, 3y-z)$ to find

the image of the vector $v = (0, 1, -1)$.

Ans: The standard basis of $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$

$\therefore e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$\therefore T(e_1) = T(1,0,0) = (2,0)$

$T(e_2) = T(0,1,0) = (1,3)$

$T(e_3) = T(0,0,1) = (0,-1)$

$\therefore$ standard matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$

Now $T(v) = Av$.

$\therefore T(0,1,-1) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\therefore T(0,1,-1) = \underline{\underline{(1,4)}}$

5. Find $T(1,-5,2)$ by using the standard matrix for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$T(x,y,z) = (2x, x+y, y+z, x+z)$

Ans: Standard basis of $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$

$\therefore e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$\therefore T(e_1) = T(1,0,0) = (2,1,0,1)$

$T(e_2) = T(0,1,0) = (0,1,1,0)$

$T(e_3) = T(0,0,1) = (0,0,1,1)$

$\therefore$ standard matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Now $T(v) = Av$

$\therefore T(1,-5,2) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -3 \\ 3 \end{bmatrix}$

$\therefore T(1,-5,2) = (2,-4,-3,3)$

Matrix of a linear transformation

Let T be a linear transformation from an n-dimensional vector space V into an m-dimensional vector space W, and let $B = \{v_1, v_2, \dots v_n\}$ be a basis for V and $C = \{w_1, w_2, \dots w_m\}$ be a basis for W. Then the matrix of T with respect to B and C is defined by

$$[T]_B^C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ where } [T(v_i)]_C = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

which is the coordinate matrix of $T(v_i)$ relative to the basis C for $i = 1, 2, \dots n$.

Problems

1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y) = (x+y, 2x-y)$. Find the matrix for T relative to the bases $B = \{(1,2), (-1,1)\}$ and $C = \{(1,0), (0,1)\}$

Ans: Let $v_1 = (1,2)$, $v_2 = (-1,1)$ and $w_1 = (1,0)$, $w_2 = (0,1)$

By definition of T, $T(v_1) = T(1,2) = (3,0)$

Let $(3,0) = c_1 w_1 + c_2 w_2$

$= c_1(1,0) + c_2(0,1) = (c_1, 0) + (0, c_2)$

$= (c_1, c_2)$

$\therefore c_1 = 3, \quad c_2 = 0$

$\therefore T(v_1) = \cancel{3(1,1)+0(1,0)}\ 3\omega_1 + 0\omega_2$

$$[T(v_1)]_C = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$T(v_2) = T(-1,1) = (0,-3)$

Let $(0,-3) = c_1\omega_1 + c_2\omega_2$

$= c_1(1,0) + c_2(0,1)$

$= (c_1, 0) + (0, c_2)$

$= (c_1, c_2)$

$\therefore c_1 = 0, \quad c_2 = -3$

$\therefore T(v_2) = 0\omega_1 - 3\omega_2$

$$[T(v_2)]_C = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$\therefore$ Matrix of T with respect to B and C is given by $[T]_B^C = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

2. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y,z) = (3x-2z, 2y-z)$. Find the matrix for T relative to the bases $B = \{(1,0,1), (1,-1,0), (0,1,1)\}$ and $C = \{(1,1), (1,0)\}$

Ans: Let $v_1 = (1,0,1)$, $v_2 = (1,-1,0)$ $v_3 = (0,1,1)$ and $\omega_1 = (1,1)$, $\omega_2 = (1,0)$

By definition of T, $T(v_1) = T(1,0,1) = (1,-1)$

Let $(1,-1) = c_1\omega_1 + c_2\omega_2$

$= c_1(1,1) + c_2(1,0)$

$= (c_1, c_1) + (c_2, 0) = (c_1 + c_2, c_1)$

$c_1 + c_2 = 1 \quad c_1 = -1$

$\therefore -1 + c_2 = 1 \Rightarrow c_2 = 1+1 = 2$

$\therefore T(v_1) = -1w_1 + 2w_2$

$$[T(v_1)]_C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$T(v_2) = T(1,-1,0) = (3,-2)$

Let $(3,-2) = c_1 w_1 + c_2 w_2$

$= c_1(1,1) + c_2(1,0)$

$= (c_1, c_1) + (c_2, 0)$

$= (c_1 + c_2, c_1)$

$\therefore c_1 + c_2 = 3, \quad c_1 = -2$

$\therefore -2 + c_2 = 3 \Rightarrow c_2 = 3+2 = 5$

$\therefore T(v_2) = -2w_1 + 5w_2$

$$[T(v_2)]_C = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$T(v_3) = T(0,1,1) = (-2,1)$

$\therefore c_1 + c_2 = -2, \quad c_1 = 1$

$\therefore 1 + c_2 = -2 \Rightarrow c_2 = -2-1 = -3$

$\therefore T(v_3) = 1w_1 - 3w_2$

$$[T(v_3)]_C = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\therefore$ Matrix of T with respect to B and C is

given by $[T]_B^C = \begin{bmatrix} -1 & -2 & 1 \\ 2 & 5 & -3 \end{bmatrix}$

Note

Let T be a linear transformation from an n-dimensional vector space V into an m-dimensional vector space W and let $B = \{v_1, v_2, \dots v_n\}$ be a basis for V and $C = \{w_1, w_2, \dots w_m\}$ be a basis for W. Then

$$[T(v)]_C = [T]_B^C [v]_B .$$

3. For the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, 2x-y)$ $B = \{(1,2), (-1,1)\}$ and $C = \{(1,0), (0,1)\}$, use the matrix of T to find $T(v)$ where $v = (5,4)$.

Ans: Let $v_1 = (1,2)$, $v_2 = (-1,1)$ and $w_1 = (1,0)$, $w_2 = (0,1)$

$T(v_1) = T(1,2) = (3,0)$

$(3,0) = c_1 w_1 + c_2 w_2$
$= c_1(1,0) + c_2(0,1)$
$= (c_1, 0) + (0, c_2)$
$= (c_1, c_2)$

$\therefore c_1 = 3 \quad c_2 = 0$

So $T(v_1) = 3w_1 + 0w_2$

$$[T(v_1)]_C = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$T(v_2) = T(-1,1) = (0,-3)$

$(0,-3) = c_1 w_1 + c_2 w_2$

$= c_1(1,0) + c_2(0,1)$

$= (c_1, 0) + (0, c_2)$

$= (c_1, c_2)$

$c_1 = 0, \quad c_2 = -3$

So $T(v_2) = 0w_1 - 3w_2$

$[T(v_2)]_C = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Matrix of T w.r.t. B and C is given by

$[T]_B^C = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

Now $v = (5,4)$

$(5,4) = c_1 v_1 + c_2 v_2$

$= c_1(1,2) + c_2(-1,1)$

$= (c_1, 2c_1) + (-c_2, c_2)$

$= (c_1 - c_2, 2c_1 + c_2)$

$c_1 - c_2 = 5$

$2c_1 + c_2 = 4$

$c_1 = 3, \quad c_2 = -2$

$\therefore (5,4) = 3v_1 - 2v_2$

$[v]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$[T(v)]_C = [T]_B^C [v]_B$

$= \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

$C = \{(1,0), (0,1)\}$ $\therefore T(v) = 9(1,0) + 6(0,1)$

i.e; $T(5,4) = (9,0) + (0,6)$

$= (9,6)$

4. Let $B = \{(-3,2), (4,-2)\}$ and $C = \{(-1,2), (2,-2)\}$ be the bases for $\mathbb{R}^2$ and let $\begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$ be the matrix for $T: \mathbb{R}^2 \to \mathbb{R}^2$ relative to B and C. Find $T(v)$, where $v = (11,-6)$.

Ans: $v_1 = (-3,2)$, $v_2 = (4,-2)$ and $w_1 = (-1,2)$

$w_2 = (2,-2)$

$v = (11,-6)$

$(11,-6) = c_1 v_1 + c_2 v_2$

$= c_1(-3,2) + c_2(4,-2)$

$= (-3c_1, 2c_1) + (4c_2, -2c_2)$

$= (-3c_1 + 4c_2, 2c_1 - 2c_2)$

$\Rightarrow -3c_1 + 4c_2 = 11$

$2c_1 - 2c_2 = -6$

$\therefore c_1 = -1, \; c_2 = 2$

$\therefore (11,-6) = -1v_1 + 2v_2$

$[v]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$[T(v)]_C = [T]_B^C [v]_B$

$= \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 17 \end{bmatrix}$

Given that $C = \{(-1,2), (2,-2)\}$

$\therefore T(v) = 16(-1,2) + 17(2,-2)$

$= (-16, 32) + (34, -34)$

$= (18, -2)$

$\therefore T(11, -6) = (18, -2)$

5. The matrix representation for $T: \mathbb{R}^2 \to \mathbb{R}^2$ is $[T]_B^B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ where $B = \{(1,1), (2,1)\}$. Find $T(x,y)$

Ans: Here B and C are same.

$v_1 = w_1 = (1,1)$, $v_2 = w_2 = (2,1)$.

$(x,y) = c_1 v_1 + c_2 v_2$

$= c_1(1,1) + c_2(2,1)$

$= (c_1, c_1) + (2c_2, c_2)$

$= (c_1 + 2c_2, c_1 + c_2)$

$\Rightarrow c_1 + 2c_2 = x$

$c_1 + c_2 = y$

$c_2 = x - y$

$\therefore c_1 + (x-y) = y \Rightarrow c_1 = y - (x-y) = y - x + y$

$= 2y - x$

$\therefore (x,y) = (2y-x)(1,1) + (x-y)(2,1)$.

$\therefore [v]_B = \begin{bmatrix} 2y - x \\ x - y \end{bmatrix}$

Now $[T(v)]_B = [T]_B^B [v]_B$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2y-x \\ x-y \end{bmatrix}$$

$$= \begin{bmatrix} 2y-x+2(x-y) \\ 3(2y-x)+4(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} 2y-x+2x-2y \\ 6y-3x+4x-4y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ x+2y \end{bmatrix}$$

Given $B = \{(1,1), (2,1)\}$

$\therefore T(x,y) = x(1,1) + (x+2y)(2,1)$

$= (x, x) + (2(x+2y), x+2y)$

$= (x+2(x+2y), x+x+2y)$

$= (x+2x+4y, 2x+2y)$

$= (3x+4y, 2x+2y)$