MODULE-3

Vector length

The length or norm of a vector $V=(V_1, V_2, \dots V_n)$ in $\mathbb{R}^n$ is $\|V\| = \sqrt{V_1^2+V_2^2+\dots+V_n^2}$.

If $\|V\|=1$, then the vector V is called a unit vector.

Note

1) $\|V\| \geq 0$ for all $V \in \mathbb{R}^n$
2) $\|V\|=0$ if and only if $V=0$.
3) If V is a vector in $\mathbb{R}^n$ and c is a scalar, then $\|cV\| = |c|\,\|V\|$
4) If V is a non-zero vector in $\mathbb{R}^n$, then the unit vector in the direction of V is $u = \frac{V}{\|V\|}$
5) The unit vector in the direction opposite to that of V is $-u = -\frac{V}{\|V\|}$.

Problems

1. Find the norm of the vector $V=(-1, 2, 0, -2, 0, 4) \in \mathbb{R}^6$.

Ans: $\|V\| = \sqrt{(-1)^2+2^2+0^2+(-2)^2+0^2+4^2}$

$= \sqrt{1+4+4+16} = \sqrt{25} = \underline{\underline{5}}$

2. Find the length of the vector $V=(3,-2) \in \mathbb{R}^2$

Ans: Length $= \|V\| = \sqrt{3^2+(-2)^2} = \sqrt{9+4} = \underline{\underline{\sqrt{13}}}$

3. Determine whether $v = \left(\frac{1}{\sqrt{15}}, \frac{-2}{\sqrt{15}}, \frac{3}{\sqrt{15}}, \frac{-1}{\sqrt{15}}\right) \in \mathbb{R}^4$ is a unit vector.

Ans: $\|v\| = \sqrt{\left(\frac{1}{\sqrt{15}}\right)^2 + \left(\frac{-2}{\sqrt{15}}\right)^2 + \left(\frac{3}{\sqrt{15}}\right)^2 + \left(\frac{-1}{\sqrt{15}}\right)^2}$

$= \sqrt{\frac{1}{15} + \frac{4}{15} + \frac{9}{15} + \frac{1}{15}} = \sqrt{\frac{15}{15}} = \sqrt{1} = 1$

Since $\|v\| = 1$, v is a unit vector.

4. Find the unit vector in the direction and in the opposite direction of $v = (-1, -2)$

Ans: $\|v\| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$

unit vector in the direction of v is

$u = \frac{v}{\|v\|} = \frac{(-1, -2)}{\sqrt{5}} = \left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$

unit vector in the opposite direction of v

is $-u = -\frac{v}{\|v\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

5. Find the unit vector in the direction of $(3, -1, 2)$ and verify that this vector has length 1. Also find the unit vector in the opposite direction of $(3, -1, 2)$.

Ans: Let $v = (3, -1, 2)$, then $\|v\| = \sqrt{9+1+4} = \sqrt{14}$

unit vector in the direction of v is

$u = \frac{v}{\|v\|} = \frac{(3,-1,2)}{\sqrt{14}}$

$= \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$

Length of u is $\|u\| = \sqrt{\frac{9}{14} + \frac{1}{14} + \frac{4}{14}}$

$= \sqrt{\frac{14}{14}} = \sqrt{1} = 1$

unit vector in the opposite direction of $(3,-1,2)$

is $-u = \frac{-v}{\|v\|} = \left(\frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$

H.W

6. Find the unit vectors in the direction and in the opposite direction of $v = (3, 2, -1, 4)$

Ans: unit vector $u = \left(\frac{3}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{4}{\sqrt{30}}\right)$

unit vector in the opposite direction is

$-u = \left(\frac{-3}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-4}{\sqrt{30}}\right)$

7. Find the vector v with length 4 and in the direction of $u = (-1, 1)$

Ans: $\|u\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

unit vector in the direction of u is $w = \frac{u}{\|u\|}$

$= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

The vector v with length 4 and in the direction of $u = (-1, 1)$ is given by

$v = 4w = \left(\frac{-4}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right) = (-2\sqrt{2}, 2\sqrt{2})$

8. Find the vector v with length 3 and in the direction of $u = (0, 2, 1, -1)$

Ans: $\|u\| = \sqrt{0+4+1+1} = \sqrt{6}$

Unit vector in the direction of u is

$$W = \frac{u}{\|v\|} = \left(0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$\therefore$ The vector v with length 3 and in the direction of $u = (0, 2, 1, -1)$ is

$$V = 3w = \left(0, \frac{6}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{-3}{\sqrt{6}}\right)$$

$$= \left(0, \sqrt{6}, \frac{3}{\sqrt{6}}, \frac{-3}{\sqrt{6}}\right)$$

9. Find all values of c such that $\|c(1,2,3)\| = 1$

Ans: Given $\|c(1,2,3)\| = 1$

$\Rightarrow |c| \, \|(1,2,3)\| = 1$

$\Rightarrow |c| \sqrt{1+4+9} = 1$

$\Rightarrow |c| \sqrt{14} = 1$

$\Rightarrow |c| = \frac{1}{\sqrt{14}}$

$\therefore \quad c = \pm \frac{1}{\sqrt{14}}$

10. Find all values of c such that $\|c(-3,1,2,-1)\| = \sqrt{3}$

Ans: Given $\|c(-3,1,2,-1)\| = \sqrt{3}$

$\Rightarrow |c| \, \|(-3,1,2,-1)\| = \sqrt{3}$

$\Rightarrow |c| \sqrt{9+1+4+1} = \sqrt{3}$

$\Rightarrow |c| \sqrt{15} = \sqrt{3}$

$\Rightarrow |c| = \frac{\sqrt{3}}{\sqrt{15}} = \frac{\sqrt{3}}{\sqrt{3} \times \sqrt{5}} = \frac{1}{\sqrt{5}}$

$\therefore c = \pm \frac{1}{\sqrt{5}}$

11. Find the norm of $u+v$, where $u=(1,-1,3,2)$ and $v=(-1,1,0,2)$

Ans: $u+v = (0,0,3,4)$

$\therefore \|u+v\| = \sqrt{0+0+9+16} = \sqrt{25} = 5$

12. Find the vector v with length 3 and in the direction of $u=(1,2,2)$

Ans: $\|u\| = \sqrt{1+4+4} = \sqrt{9} = 3$

unit vector in the direction of u is $w = \frac{u}{\|u\|}$

$= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\therefore$ The vector v with length 3 and in the direction of $u=(1,2,2)$ is $v = 3w$

$= \left(\frac{3}{3}, \frac{6}{3}, \frac{6}{3}\right)$

$= (1, 2, 2)$

Dot product and Angle

Let $u = (u_1, u_2, \dots u_n)$, $v = (v_1, v_2, \dots v_n) \in \mathbb{R}^n$.

Then dot product of u and v is defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n .$$

Note

$u \cdot v$ is a scalar.

Properties of dot product.

If u, v and w are vectors in $\mathbb{R}^n$ and c is a scalar, then

1) $u \cdot v = v \cdot u$
2) $u \cdot (v + w) = u \cdot v + u \cdot w$
3) $u \cdot (cv) = c(u \cdot v) = (cu) \cdot v$
4) $v \cdot v = \|v\|^2$
5) $v \cdot v = 0$ if and only if $v = 0$.

Angle between two vectors in $\mathbb{R}^n$.

The angle θ between two non zero vectors u and v in $\mathbb{R}^n$ is defined by

$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}, \quad 0 \le \theta \le \pi.$$

Note

If u and v are orthogonal vectors in $\mathbb{R}^n$ then

$$u \cdot v = 0$$

Cauchy Schwarz inequality

If u and v are vectors in $\mathbb{R}^n$, then

$\boxed{|u\cdot v| \le \|u\| \, \|v\|.}$ where $|u\cdot v|$ denotes the absolute value of $u\cdot v$

Problems

1. Find the dot product of $u=(4,2,1,-1)$ and $v=(3,-1,4,2)$.

Ans: $u\cdot v = (4\times 3)+(2\times -1)+(1\times 4)+(-1\times 2)$

$= 12-2+4-2 = \underline{\underline{12}}$

2. Find the dot product of $u=(3,1,-2)$ and $v=(-3,2,4)$.

Ans: $u\cdot v = (3\times -3)+(1\times 2)+(-2\times 4)$

$= -9+2-8 = \underline{\underline{-15}}$

3. Consider $u, v \in \mathbb{R}^n$ such that $u\cdot u = 38$, $u\cdot v = -3$ and $v\cdot v = 79$. Find $(u+2v)\cdot(3u+v)$.

Ans: $(u+2v)\cdot(3u+v) = u\cdot(3u+v) + 2v\cdot(3u+v)$

$= (u\cdot 3u)+(u\cdot v)+(2v\cdot 3u)+(2v\cdot v)$

$= 3(u\cdot u)+(u\cdot v)+6(v\cdot u)+2(v\cdot v)$

$= 3(u\cdot u)+(u\cdot v)+6(u\cdot v)+2(v\cdot v)$

$= 3(u\cdot u)+7(u\cdot v)+2(v\cdot v)$

$= 3\times 38 + 7\times -3 + 2\times 79$

$= 251$ //

4. Consider $u, v \in \mathbb{R}^n$ such that $\|u\|^2 = 2$, $(u+v)\cdot(u-2v) = 1$ and $\|v\|^2 = 3$. Find $u \cdot v$

Ans: $(u+v)\cdot(u-2v) = 1$

$\Rightarrow (u\cdot u) - (u \cdot 2v) + (v \cdot u) - (v \cdot 2v) = 1$

$\Rightarrow \|u\|^2 - 2(u\cdot v) + (u \cdot v) - 2(v\cdot v) = 1$

$\Rightarrow \|u\|^2 - (u\cdot v) - 2\|v\|^2 = 1$

$\Rightarrow 2 - (u\cdot v) - 2\times 3 = 1$

$\Rightarrow 2 - (u\cdot v) - 6 = 1$

$\Rightarrow u\cdot v = 2 - 6 - 1 = 2 - 7 = \underline{\underline{-5}}$

5. Verify the cauchy schwarz inequality for $u = (1, 4)$ and $v = (3, -2)$.

Ans: $u\cdot v = 1\times 3 + 4 \times -2$

$= 3 - 8 = -5$

$\therefore |u\cdot v| = |-5| = 5$

$\|u\| = \sqrt{1+16} = \sqrt{17}$

$\|v\| = \sqrt{9+4} = \sqrt{13}$

$\|u\|\cdot\|v\| = \sqrt{17}\times\sqrt{13} = \sqrt{221} = 14.86$

$\therefore |u\cdot v| \leq \|u\|.\|v\|$

$\therefore$ Cauchy schwarz inequality is verified.

6. Verify Cauchy Schwarz inequality for $u=(1,1,-2)$ and $v=(1,-3,2)$.

Ans: $u \cdot v = 1\times1 + 1\times-3 + -2\times2 = 1-3-4 = -6$

$\therefore |u \cdot v| = 6$

$\|u\| = \sqrt{1+1+4} = \sqrt{6}$, $\|v\| = \sqrt{1+9+4} = \sqrt{14}$

$\therefore \|u\|\|v\| = \sqrt{6}\times\sqrt{14} = \sqrt{84} = 9.16$

$\therefore |u \cdot v| \leq \|u\|\|v\|$

Hence the result

7. Verify Cauchy Schwarz inequality for $u=(-4,0,2,-2)$ and $v=(2,0,-1,1)$

Ans: $u \cdot v = -8 + 0 + -2 + -2 = -12$

$\therefore |u \cdot v| = |-12| = 12$

$\|u\| = \sqrt{16+0+4+4} = \sqrt{24}$

$\|v\| = \sqrt{4+0+1+1} = \sqrt{6}$

$\therefore \|u\|\|v\| = \sqrt{24}\times\sqrt{6} = \sqrt{144} = 12$

$\therefore |u \cdot v| \leq \|u\|\|v\|$

Hence the inequality is verified.

8. Find the angle between the vectors $u=(1,1,1)$ and $v=(2,1,-1)$.

Ans: $\cos\theta = \dfrac{u \cdot v}{\|u\|\|v\|} = \dfrac{2+1+-1}{\sqrt{1+1+1}\sqrt{4+1+1}} = \dfrac{2}{\sqrt{3}\cdot\sqrt{6}} = \dfrac{2}{\sqrt{18}}$

$$= \frac{2}{\sqrt{2\times 9}} = \frac{\sqrt{2}}{3}$$

$$\therefore \theta = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

9. Find the angle between the vectors $u=(3,1)$ and $v=(-1,2)$

Ans: $\cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|} = \frac{-3+2}{\sqrt{9+1}\sqrt{1+4}} = \frac{-1}{\sqrt{10}\cdot\sqrt{5}} = \frac{-1}{\sqrt{50}}$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{50}}\right)$$

10. Find the angle between the vectors $u=(1,-1,0,1)$ and $v=(-1,2,-1,0)$

Ans: $\cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|} = \frac{-1-2+0+0}{\sqrt{1+1+1}\sqrt{1+4+1}} = \frac{-3}{\sqrt{18}}$

$$= \frac{-3}{\sqrt{2\times 9}}$$

$$= \frac{-3}{\sqrt{2}\times 3} = \frac{-1}{\sqrt{2}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

11. Find all vectors in R^2 that are orthogonal to $u=(2,1)$

Ans: Let $v=(v_1,v_2)$ be orthogonal to u. Then

$u\cdot v=0$

$\Rightarrow (2,1)\cdot(v_1,v_2)=0$

$\Rightarrow 2v_1+1v_2=0$

$\Rightarrow v_2 = -2v_1$

$\therefore$ ~~The~~ Every vector that is orthogonal to $(2,1)$ is of the form $v=(k,-2k)$

$= k(1,-2)$, where k is any real number.

ie; all vectors orthogonal to $u=(2,1)$ is the straight line $\underline{\underline{y=-2x}}$

12. Find all vectors in $\mathbb{R}^3$ that are orthogonal to $u=(2,-1,1)$.

Ans: Let $v=(v_1,v_2,v_3)$ be orthogonal to u. Then

$u\cdot v=0$

$\Rightarrow (2,-1,1)\cdot(v_1,v_2,v_3)=0$

$\Rightarrow 2v_1 - v_2 + v_3 = 0$

$\Rightarrow v_3 = v_2 - 2v_1$

we choose $v_1=a$ and $v_2=b$ as arbitrary numbers

so every vector that is orthogonal to $(2,-1,1)$ is of the form $v=(a,b,b-2a)$, $a,b\in\mathbb{R}$

$\therefore$ all the vectors orthogonal to $u=(2,-1,1)$ is the plane $2x-y+z=0$.

13. Find all vectors in $\mathbb{R}^3$ that are orthogonal to $u=(4,-1,0)$.

Ans: Let $v=(v_1,v_2,v_3)$ be orthogonal to u. Then

$u\cdot v=0$

$\Rightarrow (4,-1,0)\cdot(v_1,v_2,v_3)=0$

$\Rightarrow 4v_1 - v_2 = 0$

$\Rightarrow v_2 = 4v_1$ and v_3 has no restriction.

we choose $v_1 = a$ and $v_3 = b$ as arbitrary numbers.

So every vector that is orthogonal to $(4,-1,0)$ is of the form $v = (a, 4a, b)$, $a, b \in \mathbb{R}$.

$\therefore$ all the vectors orthogonal to $u = (4,-1,0)$ is the plane $4x - y = 0$, $z \in \mathbb{R}$.

14. Find all vectors in $\mathbb{R}^4$ that are orthogonal to $u = (1,-2,3,-4)$

Ans: Let $v = (v_1, v_2, v_3, v_4)$ be orthogonal to u. Then

$$u \cdot v = 0 \implies (1,-2,3,-4)\cdot(v_1, v_2, v_3, v_4) = 0$$

$$\implies v_1 - 2v_2 + 3v_3 - 4v_4 = 0$$

$$\implies v_1 = 2v_2 - 3v_3 + 4v_4$$

we choose $v_2 = a$, $v_3 = b$ and $v_4 = c$ as arbitrary numbers. So every vector that is orthogonal to $(1,-2,3,-4)$ is of the form

$$v = (2a - 3b + 4c, a, b, c), \quad a, b, c \in \mathbb{R}.$$

Inner Product.

An inner product $\langle \cdot , \cdot \rangle$ on a vector space V is a function from $V \times V \to \mathbb{R}$ satisfying following conditions.

1) $\langle u, u \rangle \geq 0$ for all $u \in V$
2) $\langle u, u \rangle = 0$ if and only if $u = 0$
3) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$
4) $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{R}$.
5) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.

Inner Product space

A vector space V together with an inner product $\langle \cdot , \cdot \rangle$ is called an inner product space.

Problems

1) Show that dot product defined on $\mathbb{R}^n$ is an inner product.

Ans: Let $u = (u_1, u_2, \ldots u_n)$, $v = (v_1, v_2, \ldots v_n) \in \mathbb{R}^n$.

The dot product of u and v is

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

1) $\langle u, u \rangle = u_1 u_1 + u_2 u_2 + \cdots + u_n u_n$

$$= u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0.$$

2) $\langle u, u \rangle = 0 \iff u_1^2 + u_2^2 + \cdots + u_n^2 = 0$

$$\iff u_1 = 0, \ u_2 = 0, \ \ldots u_n = 0.$$

$$\iff u = 0$$

3) $\langle u+v, w\rangle = (u+v)\cdot w$

$= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n)\cdot (w_1, w_2, \cdots w_n)$

$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \cdots + (u_n+v_n)w_n$

$= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + \cdots + u_nw_n + v_nw_n$

$= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n)$

$= \langle u, w\rangle + \langle v, w\rangle$.

4) $\langle \lambda u, v\rangle =$ ~~$\lambda u_1 + v_1 + \cdots$~~

$= \lambda u\cdot v$

$= (\lambda u_1, \lambda u_2, \cdots \lambda u_n)\cdot(v_1, v_2, \cdots v_n)$

$= (\lambda u_1)v_1 + (\lambda u_2)v_2 + \cdots + (\lambda u_n)v_n$

$= \lambda(u_1v_1 + u_2v_2 + \cdots + u_nv_n)$

$= \lambda\langle u, v\rangle$

5) $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

$= v_1u_1 + v_2u_2 + \cdots + v_nu_n$

$= \langle v, u\rangle$.

$\therefore$ Dot product is an inner product on $\mathbb{R}^n$.

The inner product $\langle u, v\rangle = u\cdot v$ is called

Euclidean inner product for $\mathbb{R}^n$:

2. Check whether $\langle u,v\rangle = u_1v_1 + 3u_2v_2$ is an inner product on $\mathbb{R}^2$ where $u=(u_1,u_2)$ and $v=(v_1,v_2)$.

Ans: 1) $\langle u,u\rangle = u_1^2 + 3u_2^2 \geq 0$

2) $\langle u,u\rangle = 0 \Leftrightarrow u_1^2 + 3u_2^2 = 0$

$\Leftrightarrow u_1 = 0,\ u_2 = 0$

$\Leftrightarrow u = 0$

3) $\langle u+v, w\rangle = (u_1+v_1)w_1 + 3(u_2+v_2)w_2$

$= u_1w_1 + v_1w_1 + 3u_2w_2 + 3v_2w_2$

$= (u_1w_1 + 3u_2w_2) + (v_1w_1 + 3v_2w_2)$

$= \langle u,w\rangle + \langle v,w\rangle$

4) $\langle \lambda u, v\rangle = (\lambda u_1)v_1 + 3(\lambda u_2)v_2$

$= \lambda(u_1v_1 + 3u_2v_2)$

$= \lambda\langle u,v\rangle$

5) $\langle u,v\rangle = u_1v_1 + 3u_2v_2$

$= v_1u_1 + 3v_2u_2$

$= \langle v,u\rangle$

$\therefore \langle u,v\rangle = u_1v_1 + 3u_2v_2$ is an inner product on $\mathbb{R}^2$.

3. Check whether $\langle u,v\rangle = u_1v_1 - 3u_2v_2 + u_3v_3$ is an inner product on $\mathbb{R}^3$ where $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$.

Ans: 1) $\langle u, u\rangle = u_1^2 - 3u_2^2 + u_3^2$.

Let $u = (1, 2, -1)$, then

$$\langle u, u\rangle = 1^2 - 3\times 4 + 1$$
$$= 1 - 12 + 1 = -10 < 0$$

So axiom 1 is not satisfied.

$\langle u, v\rangle = u_1v_1 - 3u_2v_2 + u_3v_3$ is not an inner product on R^3.

4. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in M_{2,2}$.

Show that the function

$\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$ is an inner product on $M_{2,2}$.

Ans: 1) $\langle A, A\rangle = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$.

2) $\langle A, A\rangle = 0 \Leftrightarrow a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = 0$

$$\Leftrightarrow a_{11} = a_{12} = a_{21} = a_{22} = 0$$
$$\Leftrightarrow A = 0$$

3) $\langle A+B, C\rangle = (a_{11}+b_{11})c_{11} + (a_{12}+b_{12})c_{12} + (a_{21}+b_{21})c_{21} + (a_{22}+b_{22})c_{22}$

$= a_{11}c_{11} + b_{11}c_{11} + a_{12}c_{12} + b_{12}c_{12} + a_{21}c_{21} + b_{21}c_{21} + a_{22}c_{22} + b_{22}c_{22}$

$= a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + a_{22}c_{22} + b_{11}c_{11} + b_{12}c_{12}$

$+ b_{21}c_{21} + b_{22}c_{22}$

$= \langle A, C\rangle + \langle B, C\rangle$

4) $\langle \lambda A, B\rangle = \lambda a_{11}b_{11} + \lambda a_{12}b_{12} + \lambda a_{21}b_{21} + \lambda a_{22}b_{22}$

$= \lambda(a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22})$

$= \lambda\langle A, B\rangle$

5) $\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

$= b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22}$

$= \langle B, A\rangle$.

$\therefore$ ~~Given~~ $\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

is an inner product on $M_{2,2}$.

Note

1) $\langle u, 0\rangle = \langle 0, u\rangle = 0$

2) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

3) $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$.

Length, Distance and Angle in an inner product space

Let u and v be vectors in an inner product space V.

1) The length or norm of u induced by an inner product is $\|u\| = \sqrt{\langle u, u\rangle}$

2) The distance between u and v is defined by $d(u,v) = \|u - v\|$

3) The angle between two non zero vectors u and v is defined by

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}$$

4) u and v are said to be orthogonal if $\langle u, v\rangle = 0$.

Cauchy-Schwarz inequality in an inner product space.

Let u and v be vectors in an inner product space V, then $|\langle u, v\rangle| \leq \|u\|\,\|v\|$ is known as Cauchy Schwarz inequality in an inner product space.

Problems

1) Let $u = (4, 2)$ and $v = (2, -2)$ be vectors in $\mathbb{R}^2$ with the inner product $\langle u, v\rangle = u_1 v_1 + 2u_2 v_2$

(i) Show that u and v are orthogonal.

(ii) Find the angle b/w u and v with respect to Euclidean inner product.

Ans: (i) $u = (4, 2)$ and $v = (2, -2)$

$\therefore \langle u, v\rangle = u_1 v_1 + 2u_2 v_2$

$\Rightarrow \langle u, v\rangle = 4\times 2 + 2\times 2\times -2 = 8 - 8 = 0$

Since $\langle u, v\rangle = 0$, u and v are orthogonal.

(1) Euclidean inner product is $\langle u,v\rangle = u\cdot v$

$u=(4,2) \quad v=(2,-2)$

$\therefore \langle u,v\rangle = u\cdot v = 8-4=4$

$\|u\| = \sqrt{4^2+2^2} = \sqrt{16+4} = \sqrt{20}$

$\|v\| = \sqrt{2^2+(-2)^2} = \sqrt{4+4} = \sqrt{8}$

$\therefore \cos\theta = \dfrac{\langle u,v\rangle}{\|u\|\,\|v\|} = \dfrac{4}{\sqrt{20}\sqrt{8}} = \dfrac{4}{\sqrt{160}}$

$= \dfrac{4}{\sqrt{16\times 10}}$

$= \dfrac{4}{4\sqrt{10}} = \dfrac{1}{\sqrt{10}}$

$\therefore \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$

2) Find the angle between $u=(0,1,-2)$ and $v=(3,-2,1)$ with respect to the inner product

$\langle u,v\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$.

Ans: $\langle u,v\rangle = (0\times 3) + (2\times 1\times -2) + (3\times -2\times 1)$

$= 0-4-6 = -10$

$\cancel{\|u\| = \sqrt{0^2+1^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}}$

$\cancel{\|v\| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{9+4+1} = \sqrt{14}}$

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1u_1 + 2u_2u_2 + 3u_3u_3}$

$= \sqrt{0+2+12} = \sqrt{14}$

$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{v_1v_1 + 2v_2v_2 + 3v_3v_3}$

$= \sqrt{9+8+3} = \sqrt{20}$

$\therefore \cos\theta = \dfrac{-10}{\sqrt{14}\sqrt{20}} = \dfrac{-10}{\sqrt{14}\cdot 2\sqrt{5}} = \dfrac{-5}{\sqrt{70}}$

$$\therefore \theta = \cos^{-1}\left(\frac{-5}{\sqrt{70}}\right)$$

3. Find the distance between $u = (4,2)$ and $v = (2,-2)$ with respect to the inner product $\langle u,v\rangle = u_1 v_1 + 2u_2 v_2$. and the Euclidean inner product.

Ans: $d(u,v) = \|u-v\| = \sqrt{\langle u-v, u-v\rangle}$

$u - v = (4-2, 2--2) = (2,4)$

$\langle u-v, u-v\rangle = \langle (2,4), (2,4)\rangle$

$= 2\times 2 + 2\times 4\times 4$

$= 4 + 32 = 36$

$\therefore d(u,v) = \sqrt{36} = \underline{\underline{6}}$

Euclidean inner product is $\langle u,v\rangle = u\cdot v$

$= u_1 v_1 + u_2 v_2$

$\therefore \langle u-v, u-v\rangle = \langle (2,4), (2,4)\rangle$

$= 2\times 2 + 4\times 4$

$= 4 + 16 = 20$

$\therefore d(u,v) = \sqrt{20} = \underline{\underline{2\sqrt{5}}}$

H.W

4. Find the distance between $u = (0,1,-2)$ and $v = (3,-2,1)$ with respect to the inner product $\langle u,v\rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$.

Ans: $d(u,v) = \|u-v\| = \sqrt{\langle u-v, u-v\rangle}$

$u - v = (0-3, 1-\overline{2}, -2-1) = (-3, 3, -3)$

$\langle u-v, u-v\rangle = \langle (-3,3,-3), (-3,3,-3)\rangle$

$= -3\times-3 + 2\times3\times3 + 3\times-3\times-3$

$= 9 + 18 + 27 = 54$

$\therefore d(u,v) = \sqrt{54}$

$= \sqrt{9\times6} = \underline{\underline{3\sqrt{6}}}$

5. Verify the Cauchy Schwarz inequality for $u = (0,1,0)$, $v = (1,0,-1)$ with inner product

$\langle u,v\rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3$

Ans: $\langle u,v\rangle = 0 + 0 + 0 = 0$

$\therefore |\langle u,v\rangle| = 0$ — ①

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{0+2+0} = \sqrt{2}$

$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{1+0+1} = \sqrt{2}$

$\therefore \|u\|\,\|v\| = \sqrt{2}.\sqrt{2} = 2$ — ②

From ① & ② $|\langle u,v\rangle| \le \|u\|\,\|v\|$

Hence the result.

6. Verify the Cauchy Schwarz inequality for $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$ with inner product

$\langle A,B\rangle = a_{11} b_{11} + a_{12} b_{12} + a_{21} b_{21} + a_{22} b_{22}$

Ans: $|\langle A,B\rangle| = |0\times1 + 1\times1 + 2\times2 + -1\times-2|$

$= |0+1+4+2| = |7| = 7$

$\|A\| = \sqrt{\langle A,A\rangle} = \sqrt{0+1+4+1} = \sqrt{6}$

$\|B\| = \sqrt{\langle B,B\rangle} = \sqrt{1+1+4+4} = \sqrt{10}$

$||A|| \, ||B|| = \sqrt{6} \cdot \sqrt{10} = \sqrt{60}$

$\therefore |\langle A, B \rangle| \leq ||A|| \, ||B||$

$\therefore$ Cauchy Schwarz inequality is verified.

Orthogonal Projection

If u and v are two vectors in a vector space, then the orthogonal projection of u onto v is

$$\text{Proj}_v u = \left(\frac{u \cdot v}{v \cdot v}\right) v \quad , \text{ where } v \neq 0.$$

Similarly if u and v are vectors in an inner product space V, such that $v \neq 0$, then the orthogonal projection of u onto v is

$$\text{Proj}_v u = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$$

Problems

1) Find the orthogonal projection of $u = (6, 2, 4)$ onto $v = (1, 2, 0)$ in $\mathbb{R}^3$.

HW

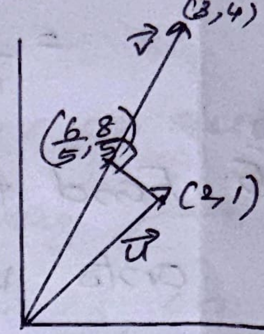
Ans: $\text{Proj}_v u = \left(\frac{u \cdot v}{v \cdot v}\right) v$

$$= \left(\frac{6 + 4 + 0}{1 + 4 + 0}\right)(1, 2, 0)$$

$$= \left(\frac{10}{5}\right)(1, 2, 0) = 2(1, 2, 0) = (2, 4, 0).$$

2. Find the orthogonal projection of $u = (2, 1)$ onto $v = (3, 4)$ in $\mathbb{R}^2$.

Ans: $\text{Proj}_v u = \left(\frac{u\cdot v}{v\cdot v}\right) v$



$= \left(\frac{6+4}{9+16}\right)(3,4)$

$= \frac{10}{25}(3,4)$

$= \left(\frac{30}{25}, \frac{40}{25}\right) = \underline{\left(\frac{6}{5}, \frac{8}{5}\right)}$

3. Find the orthogonal projection of $u=(0,1,3,-6)$ onto $v=(-1,1,2,2)$ and v onto u in $\mathbb{R}^4$.

Ans: $\text{Proj}_v u = \left(\frac{u\cdot v}{v\cdot v}\right) v$

$= \left(\frac{0+1+6-12}{1+1+4+4}\right)(-1,1,2,2)$

$= \left(\frac{-5}{10}\right)(-1,1,2,2) = \frac{-1}{2}(-1,1,2,2)$

$= \underline{\left(\frac{1}{2}, \frac{-1}{2}, -1, -1\right)}$

$\text{Proj}_u v = \left(\frac{v\cdot u}{u\cdot u}\right) u$

$= \left(\frac{0+1+6-12}{0+1+9+36}\right)(0,1,3,-6)$

$= \frac{-5}{46}(0,1,3,-6)$

$= \left(0, \frac{-5}{46}, \frac{-15}{46}, \frac{30}{46}\right) = \underline{\left(0, \frac{-5}{46}, \frac{-15}{46}, \frac{15}{23}\right)}$

4. Find the orthogonal projection of $u=(2,1)$ onto $v=(3,4)$ with inner product $\langle u,v\rangle = u_1v_1 + 0.25u_2v_2$ in $\mathbb{R}^2$.

Ans: $\text{Proj}_v u = \frac{\langle u,v\rangle}{\langle v,v\rangle} v = \frac{2\times 3 + 0.25\times 1\times 4}{3\times 3 + 0.25\times 4\times 4}(3,4)$

$= \frac{7}{13}(3,4) = \left(\frac{21}{13}, \frac{28}{13}\right)$

H.W

5. Find the orthogonal projection of $u=(6,2,4)$ onto $v=(1,2,0)$ with inner product $\langle u,v\rangle = u_1v_1 + 2u_2v_2 + u_3v_3$ in $\mathbb{R}^3$.

Ans: $\text{Proj}_v u = \frac{\langle u,v\rangle}{\langle v,v\rangle} v$

$$= \frac{6 + 4\times 2 + 0}{1 + 8 + 0}(1,2,0)$$

$$= \frac{14}{9}(1,2,0) = \left(\frac{14}{9}, \frac{28}{9}, 0\right)$$

6. Let $u=(4,2)$ and $v=(3,-2)$ be vectors in $\mathbb{R}^2$. Find the orthogonal projection of v onto u under the inner product $\langle u,v\rangle = u_1v_1 + 2u_2v_2$

Ans: $\text{Proj}_u v = \frac{\langle u,v\rangle}{\langle u,u\rangle} u$

$$= \frac{12 + -8}{16 + 8}(4,2)$$

$$= \frac{4}{24}(4,2) = \frac{1}{6}(4,2)$$

$$= \left(\frac{4}{6}, \frac{2}{6}\right) = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Orthogonal and orthonormal sets

A set S of vectors in an inner product space V is called orthogonal if every pair of distinct vectors in S is orthogonal. If, in addition every vector in S is a unit vector then S is called orthonormal.

Orthonormal basis

If an orthonormal set is a basis for a vector space, then it is called an orthonormal basis.

~~Problems~~

Note

If $S = \{v_1, v_2, \dots v_n\}$ is an orthogonal set of non zero vectors in an inner product space V, then the set S is linearly independent.

Problems

1. Show that $S = \{v_1, v_2, v_3\}$ where $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ $v_2 = \left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)$, $v_3 = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ is an orthonormal set in $\mathbb{R}^3$. Also show that S is an orthonormal basis for $\mathbb{R}^3$.

Ans: $v_1 \cdot v_2 = \frac{-1}{6} + \frac{1}{6} + 0 = 0$

$$v_1 \cdot v_3 = \frac{1}{\sqrt{2}} \times \frac{2}{3} + \frac{1}{\sqrt{2}} \times \frac{-2}{3} + 0 = 0$$

$$v_2 \cdot v_3 = -\frac{\sqrt{2}}{36} \cdot \frac{2}{3} + \frac{\sqrt{2}}{36} \cdot \frac{-2}{3} + \frac{2\sqrt{2}}{3} \cdot \frac{1}{3}$$

$$= -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = \frac{-2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0.$$

Since every pair of distinct vectors are orthogonal, $S = \{v_1, v_2, v_3\}$ is orthogonal.

$$\|v_1\| = \sqrt{\frac{1}{2}+\frac{1}{2}+0} = \sqrt{1} = 1$$

$$\|v_2\| = \sqrt{\frac{2}{36}+\frac{2}{36}+\frac{8}{9}} = \sqrt{\frac{18}{18}} = \sqrt{1} = 1$$

$$\|v_3\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = \sqrt{\frac{9}{9}} = \sqrt{1} = 1$$

$\therefore$ Every vectors in S are unit vectors.

$\therefore S$ is an orthonormal set.

$\therefore$ S is orthogonal set of non zero vectors.
so S is linearly independent and hence S is an orthonormal basis.

2. Determine whether $B = \{(\sin\theta, \cos\theta), (\cos\theta, -\sin\theta)\}$ is an orthonormal basis for $\mathbb{R}^2$.

Ans: Let $B = \{v_1, v_2\}$ with $v_1 = (\sin\theta, \cos\theta)$ and
$v_2 = (\cos\theta, -\sin\theta)$.

$$v_1 \cdot v_2 = \sin\theta\cos\theta - \cos\theta\sin\theta = 0$$

$\therefore$ B is an orthogonal set.

$$\|v_1\| = \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1$$

$$\|v_2\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

Since both vectors are unit vectors also, B is an orthonormal set.

$\therefore$ B is orthogonal set of non zero vectors
so B is linearly independent and hence B is an orthonormal basis.

Coordinate matrix

If $B = \{v_1, v_2, \dots v_n\}$ is an orthonormal basis for an inner product space V, then the coordinate matrix of a vector u relative to B is

$$[u]_B = \begin{bmatrix} \langle u, v_1 \rangle \\ \langle u, v_2 \rangle \\ \vdots \\ \langle u, v_n \rangle \end{bmatrix}$$

Problems

1) Find the coordinate matrix of $u = (5, -5, 2)$ relative to the orthonormal basis

$$B = \left\{ \left(\frac{3}{5}, \frac{4}{5}, 0\right), (0,0,1), \left(\frac{-4}{5}, \frac{3}{5}, 0\right) \right\}$$

Ans: Let $v_1 = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$ $v_2 = (0,0,1)$

$v_3 = \left(\frac{-4}{5}, \frac{3}{5}, 0\right)$

$$\langle u, v_1 \rangle = u \cdot v_1 = (5,-5,2) \cdot \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

$$= 5 \times \frac{3}{5} + -5 \times \frac{4}{5} + 2 \times 0$$

$$= 3 - 4 = -1$$

$$\langle u, v_2 \rangle = u \cdot v_2 = (5,-5,2) \cdot (0,0,1)$$

$$= 0 + 0 + 2 = 2$$

$$\langle u, v_3 \rangle = u \cdot v_3 = (5,-5,2) \cdot \left(\frac{-4}{5}, \frac{3}{5}, 0\right)$$

$$= 5 \times -\frac{4}{5} + -5 \times \frac{3}{5} + 2 \times 0$$

$$= -4 - 3 = -7$$

$\therefore$ coordinate matrix of u relative to B is

$$[u]_B = \begin{bmatrix} -1 \\ 2 \\ -7 \end{bmatrix}$$

2. Find the coordinate matrix of $u = (2, -1, 4, 3)$ relative to the orthonormal basis

$$B = \left\{ \left(\frac{5}{13}, 0, \frac{12}{13}, 0\right), (0,1,0,0), \left(\frac{12}{13}, 0, \frac{5}{13}, 0\right), (0,0,0,1) \right\}$$

Ans: Let $v_1 = \left(\frac{5}{13}, 0, \frac{12}{13}, 0\right)$ $v_2 = (0,1,0,0)$,

$v_3 = \left(\frac{12}{13}, 0, \frac{5}{13}, 0\right)$ $v_4 = (0,0,0,1)$

$$\langle u, v_1 \rangle = u \cdot v_1 = (2,-1,4,3) \cdot \left(\frac{5}{13}, 0, \frac{12}{13}, 0\right)$$

$$= \frac{10}{13} + 0 + \frac{48}{13} + 0 = \frac{58}{13}$$

$$\langle u, v_2 \rangle = u \cdot v_2 = (2,-1,4,3) \cdot (0,1,0,0)$$

$$= 0 - 1 + 0 + 0 = -1$$

$$\langle u, v_3 \rangle = u \cdot v_3 = (2,-1,4,3) \cdot \left(\frac{12}{13}, 0, \frac{5}{13}, 0\right)$$

$$= \frac{24}{13} + 0 + \frac{20}{13} + 0 = \frac{44}{13}$$

$$\langle u, v_4 \rangle = u \cdot v_4 = (2,-1,4,3) \cdot (0,0,0,1)$$

$$= 0 + 0 + 0 + 3 = 3$$

$$\therefore [u]_B = \begin{bmatrix} 58/13 \\ -1 \\ 44/13 \\ 3 \end{bmatrix}$$

HW

3. Find the coordinate matrix of $u=(1,2)$ relative to the orthonormal basis

$$B=\left\{\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right), \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)\right\}$$

Ans: Let $v_1 = \left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$ $v_2 = \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)$

$$\therefore \langle u, v_1\rangle = u\cdot v_1 = (1,2)\cdot\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$$

$$= \frac{-2}{\sqrt{13}} + \frac{6}{\sqrt{13}} = \frac{4}{\sqrt{13}}$$

$$\langle u, v_2\rangle = u\cdot v_2 = (1,2)\cdot\left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)$$

$$= \frac{3}{\sqrt{13}} + \frac{4}{\sqrt{13}} = \frac{7}{\sqrt{13}}$$

$$\therefore [u]_B = \begin{bmatrix} 4/\sqrt{13} \\ 7/\sqrt{13} \end{bmatrix}$$

Gram - Schmidt orthonormalization.

The Gram - Schmidt orthonormalization is an algorithm used to find an orthonormal basis.

Let $B=\{v_1, v_2, \ldots v_n\}$ be a basis for an inner product space V. Then consider vectors $w_1, w_2, \ldots w_n$ as follows:

$w_1 = v_1$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$w_4 = v_4 - \frac{\langle v_4, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_4, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_4, w_3\rangle}{\langle w_3, w_3\rangle} w_3$$

$$\cdots\cdots$$

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 \cdots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

Then $C = \{w_1, w_2, \ldots w_n\}$ is an orthogonal basis for V. and $D = \{u_1, u_2, \ldots u_n\}$ is an orthonormal basis, where $u_1 = \frac{w_1}{\|w_1\|}$, $u_2 = \frac{w_2}{\|w_2\|}$ ... $u_n = \frac{w_n}{\|w_n\|}$.

Problems

1) Find the orthonormal basis corresponding to the basis $B = \{(1,1), (1,0)\}$ for $\mathbb{R}^2$.

Ans: $B = \{v_1, v_2\}$, where $v_1 = (1,1)$ and $v_2 = (1,0)$

$$w_1 = v_1 = (1,1)$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= (1,0) - \frac{(1,0)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1)$$

$$= (1,0) - \frac{1}{1+1}(1,1) = (1,0) - \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$\therefore$ orthogonal basis $C = \{w_1, w_2\}$

$$= \left\{(1,1), \left(\frac{1}{2}, \frac{-1}{2}\right)\right\}$$

now $u_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{(1,1)}{\sqrt{1+1}} = \dfrac{(1,1)}{\sqrt{2}}$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$u_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{\left(\frac{1}{2}, -\frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}}} = \dfrac{\left(\frac{1}{2}, \frac{-1}{2}\right)}{\sqrt{\frac{1}{2}}}$

$$= \left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

$\therefore$ orthonormal basis $D = \{u_1, u_2\}$

$$= \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$$

HW

2. Find an orthonormal basis for the subspace of $\mathbb{R}^3$, spanned by the vectors $v_1 = (0,1,0)$ and $v_2 = (1,1,1)$.

Ans: $B = \{v_1, v_2\}$ where $v_1 = (0,1,0)$ & $v_2 = (1,1,1)$

$w_1 = v_1 = (0,1,0)$

$w_2 = v_2 - \dfrac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$$= (1,1,1) - \frac{(1,1,1)\cdot(0,1,0)}{(0,1,0)\cdot(0,1,0)}(0,1,0)$$

$$= (1,1,1) - \frac{1}{1}(0,1,0)$$

$$= (1,1,1) - (0,1,0) = (1,0,1)$$

$\therefore$ orthogonal basis $C = \{w_1, w_2\}$

$$= \{(0,1,0), (1,0,1)\}$$

Now $u_1 = \frac{w_1}{\|w_1\|} = \frac{(0,1,0)}{\sqrt{0+1+0}} = (0,1,0)$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(1,0,1)}{\sqrt{1+0+1}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$\therefore$ orthonormal basis $D = \{u_1, u_2\}$

$$= \left\{(0,1,0), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\}$$

3. Find the orthonormal basis corresponding to the basis $B = \{(1,1,0), (1,2,0), (0,1,2)\}$ for $\mathbb{R}^3$.

Ans: $B = \{V_1, V_2, V_3\}$ $V_1 = (1,1,0)$, $V_2 = (1,2,0)$

$V_3 = (0,1,2)$

$$w_1 = V_1 = (1,1,0)$$

$$w_2 = V_2 - \frac{\langle V_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= (1,2,0) - \frac{(1,2,0)\cdot(1,1,0)}{(1,1,0)\cdot(1,1,0)}(1,1,0)$$

$$= (1,2,0) - \frac{1+2}{1+1}(1,1,0)$$

$$= (1,2,0) - \left(\frac{3}{2}, \frac{3}{2}, 0\right) = \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$w_3 = V_3 - \frac{\langle V_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle V_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= (0,1,2) - \frac{(0,1,2)\cdot(1,1,0)}{(1,1,0)\cdot(1,1,0)}(1,1,0) - \frac{(0,1,2)\cdot\left(\frac{-1}{2}, \frac{1}{2}, 0\right)}{\left(\frac{-1}{2}, \frac{1}{2}, 0\right)\cdot\left(\frac{-1}{2}, \frac{1}{2}, 0\right)}\left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{\frac{1}{2}}{\frac{1}{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$= (0,1,2) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$= (0,0,2)$$

$\therefore$ orthogonal basis $C = \{w_1, w_2, w_3\}$

$$= \left\{(1,1,0), \left(\frac{-1}{2}, \frac{1}{2}, 0\right) (0,0,2)\right\}$$

Now $u_1 = \frac{w_1}{\|w_1\|} = \frac{(1,1,0)}{\sqrt{1+1}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(\frac{-1}{2}, \frac{1}{2}, 0\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+0}} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{(0,0,2)}{\sqrt{0+0+4}} = (0,0,1)$$

$\therefore$ orthonormal basis $D = \{u_1, u_2, u_3\}$

$$= \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) (0,0,1)\right\}$$

4. Find the orthonormal Set corresponding to the set $S = \{(2,-1), (-2,10)\}$ in $\mathbb{R}^2$ with the inner product $\langle u,v \rangle = 2u_1v_1 + u_2v_2$.

Ans: $S = \{v_1, v_2\}$ $v_1 = (2,-1)$ $v_2 = (-2, 10)$

$w_1 = v_1 = (2,-1)$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= (-2,10) - \frac{\langle(-2,10),(2,-1)\rangle}{\langle(2,-1),(2,-1)\rangle}(2,-1)$$

$$= (-2,10) - \frac{-8+-10}{8+1}(2,-1)$$

$$= (-2,10) - (-4,2) = (2,8)$$

$\therefore$ orthogonal basis $C = \{w_1, w_2\}$

$$= \{(2,-1), (2,8)\}$$

Now $u_1 = \frac{w_1}{\|w_1\|} = \frac{(2,-1)}{\sqrt{8+1}} = \cancel{\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)}$

$$= \left(\frac{2}{3}, \frac{-1}{3}\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(2,8)}{\sqrt{8+64}} = \frac{(2,8)}{\sqrt{72}} \cancel{= \frac{(2,8)}{\sqrt{36 \times 8}}}$$

$$= \frac{(2,8)}{\sqrt{36\times 2}} = \frac{(2,8)}{6\sqrt{2}} \cancel{(3\sqrt{2})}$$

$$= \left(\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)$$

$\therefore$ orthonormal basis $D = \{u_1, u_2, u_3\}$

$$= \left\{\left(\frac{2}{3}, \frac{-1}{3}\right)\left(\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)\right\}$$

5. Find the orthonormal set corresponding to the set $S = \{(-2,2,1,0), (1,-8,0,1\}$

Ans: $D = \left\{\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)\right\}$

Projection onto a Subspace

If $\{u_1, u_2, \dots u_n\}$ is an orthonormal basis for a subspace S of R^n, then projection of any vector $v \in R^n$ onto subspace S is denoted by $\text{Proj}_S\, v$ and is given by

$$\text{Proj}_S\, v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \dots + (v \cdot u_n)u_n$$

Problems

1) Find the orthogonal projection of the vector $v = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ onto the subspace S of R^n spanned by the vectors $w_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$

Ans: Here $w_1 \cdot w_2 = 0$. So $\{w_1, w_2\}$ is an orthogonal basis for S.

$$\therefore u_1 = \frac{w_1}{\|w_1\|} = \frac{\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}}{\sqrt{9+1}} = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}}{\sqrt{25}} = \frac{1}{5}\begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore D = \{u_1, u_2\}$ is an orthonormal basis

Now $\text{Proj}_S\, v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2$

$$= \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}\right)\begin{bmatrix} 0 \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left(0 + \frac{3}{\sqrt{10}} + \frac{3}{\sqrt{10}}\right)\begin{bmatrix}0\\ 3/\sqrt{10}\\ 1/\sqrt{10}\end{bmatrix} + (-1+0+0)\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$= \frac{6}{\sqrt{10}}\begin{bmatrix}0\\ 3/\sqrt{10}\\ 1/\sqrt{10}\end{bmatrix} + (-1)\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\ 18/10\\ 6/10\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\ 9/5\\ 3/5\end{bmatrix}$$

2. Find the orthogonal projection of the vector $v = \begin{bmatrix}1\\1\\3\\3\end{bmatrix}$ onto the subspace S of $\mathbb{R}^4$ spanned by the vectors $w_1 = \begin{bmatrix}-2\\2\\1\\0\end{bmatrix}$ and $w_2 = \begin{bmatrix}1\\-8\\0\\1\end{bmatrix}$

Ans: Here $w_1 \cdot w_2 \neq 0$. So we have to find orthogonal basis.

Let $v_1 = (-2, 2, 1, 0)$ and $v_2 = (1, -8, 0, 1)$

$\therefore w_1 = v_1 = (-2, 2, 1, 0)$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= (1,-8,0,1) - \frac{(1,-8,0,1)\cdot(-2,2,1,0)}{(-2,2,1,0)\cdot(-2,2,1,0)}(-2,2,1,0)$$

$$= (1,-8,0,1) - \frac{(-2-16)}{+4+4+1}(-2,2,1,0)$$

$$= (1,-8,0,1) - \frac{-18}{9}(-2,2,1,0)$$

$$= (1,-8,0,1) + \cancel{(-36,36,18,0)}\,(-4,4,2,0)$$

$$= (-3,-4,2,1)$$

$\therefore C = \{w_1, w_2\} = \{(-2,2,1,0), (-3,-4,2,1)\}$ is an orthogonal basis.

Now $u_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{(-2,2,1,0)}{\sqrt{4+4+1}} = \dfrac{(-2,2,1,0)}{3}$

$= \left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right)$

$u_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{(-3,-4,2,1)}{\sqrt{9+16+4+1}} = \dfrac{(-3,-4,2,1)}{\sqrt{30}}$

$= \left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$

$\therefore D = \{u_1, u_2\} = \left\{\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)\right\}$ is an orthonormal basis for S.

$\text{Proj}_S\, v = \cancel{(u \cdot v)} \; (v \cdot u_1)u_1 + (v \cdot u_2)u_2$

$= \left((1,1,3,3)\cdot\left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right)\right)\left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right)$

$+ \left((1,1,3,3)\cdot\left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)\right)\left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$

$= 1\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right) + \frac{2}{\sqrt{30}}\left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$

$= \left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right) + \left(\frac{-6}{30}, \frac{-8}{30}, \frac{4}{30}, \frac{2}{30}\right)$

$= \left(\frac{-26}{30}, \frac{12}{30}, \frac{14}{30}, \frac{2}{30}\right) = \left(\frac{-13}{15}, \frac{6}{15}, \frac{7}{15}, \frac{1}{15}\right)$

$= \begin{bmatrix} -13/15 \\ 6/15 \\ 7/15 \\ 1/15 \end{bmatrix}$

The least Squares Problem.

The least Squares regression line is a straight line that best fits a set of data points.

The best line of the form $y = a + bx$ is the one that is closest to all the given points.

Suppose $(2.5, 0.5)$, $(2,2)$ and $(1, 2.6)$ be three points in $\mathbb{R}^2$. The problem is to find the best line of the form $y = a + bx$ to fit the given data. Since the three points are not collinear, the best line is the one that is closest to all the three points. To quantify 'best' we need to measure the error, which we define as the sum of the squares of the vertical distances of the points to the line.

$\therefore$ we have to find particular values $a = a_0$ and $b = b_0$ that minimize this error. Then $y = a_0 + b_0 x$ is the line which is best fit to the data.

If there is a line passing through all the three points, then we have a linear system,

$a_0 + b_0 2.5 = 0.5$

$a_0 + b_0 2 = 2$

$a_0 + b_0 \cdot 1 = 2.6$. Since there is no such line, above system is inconsistent. The matrix representation of the system $Ax=b$ where

$$A = \begin{bmatrix} 1 & 2.5 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5 \\ 2 \\ 2.6 \end{bmatrix}$$

ie; we seek the vector $x = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ such that sum of squares of vertical distances of the points to the line is minimized. Minimizing this is equivalent to minimize $||Ax-b||$. Using linear algebraic techniques, it is proven that the x for which $||Ax-b||$ minimized is the solution of

$$A^T(Ax-b)=0$$

$$\Rightarrow A^TAx - A^Tb = 0$$

$$\Rightarrow A^TAx = A^Tb$$

$A^TAx = A^Tb$ is called the normal equations of the least squares problem $Ax=b$.

$\therefore$ least square solution of problem $Ax=b$ is same as the solution of its normal equations.

Problems

1. Find the best fit straight line closest to three non collinear points $(1,0)$, $(2,1)$ and $(3,3)$.

Ans: Let $y = a_0 + b_0 x$ be the best fit line.

Then a_0 and b_0 are given by the least square solution of $Ax=b$, where

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

By solving its normal equations $A^TAX = A^Tb$, we get X.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore A^TAX = A^Tb \Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

To solve this we use Gauss Elimination method

Augmented matrix is $\begin{bmatrix} 3 & 6 & 4 \\ 6 & 14 & 11 \end{bmatrix} \quad R_1 \to \frac{R_1}{3}$

$$\sim \begin{bmatrix} 1 & 2 & 4/3 \\ 6 & 14 & 11 \end{bmatrix} \quad R_2 \to R_2 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 4/3 \\ 0 & 2 & 3 \end{bmatrix} \quad R_2 \to \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & 2 & 4/3 \\ 0 & 1 & 3/2 \end{bmatrix}$$

$\therefore$ ~~Rank~~ $a_0 + 2b_0 = 4/3$

$b_0 = 3/2$

$\therefore a_0 + 2 \cdot \frac{3}{2} = \frac{4}{3} \Rightarrow a_0 = \frac{4}{3} - 3 = -5/3$

$\therefore$ The best fit st. line is $y = \frac{-5}{3} + \frac{3}{2}x$

2. Find the least square solution of the system $Ax = b$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$

Ans: we can find x by solving the normal equations

$A^T A x = A^T b$

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore A^T A x = A^T b \Rightarrow \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which can be solved using Gauss Elimination method.

Augmented matrix $= \begin{bmatrix} 6 & 5 & 1 \\ 5 & 6 & -1 \end{bmatrix}$ $\quad R_1 \to \frac{R_1}{6}$

$\sim \begin{bmatrix} 1 & 5/6 & 1/6 \\ 5 & 6 & -1 \end{bmatrix}$ $\quad R_2 \to R_2 - 5R_1$

$\sim \begin{bmatrix} 1 & 5/6 & 1/6 \\ 0 & 11/6 & -11/6 \end{bmatrix}$ $\quad R_2 \to R_2 \times \frac{6}{11}$

$\sim \begin{bmatrix} 1 & 5/6 & 1/6 \\ 0 & 1 & -1 \end{bmatrix}$

$\therefore a_0 + \frac{5}{6} b_0 = \frac{1}{6}$ and $b_0 = -1$

$\therefore a_0 - \frac{5}{6} = \frac{1}{6} \Rightarrow a_0 = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$

$\therefore$ Least square solution is $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. Find the least squares regression line for the following data

x	1	2	3	4	5
y	1	5	3	6	9

Ans: Let $y = a_0 + b_0 x$ be the least square regression line.

Then a_0 and b_0 are given by the least square solution of $Ax = b$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 6 \\ 9 \end{bmatrix}$

By solving its normal equation $A^T A X = A^T b$, we get X.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 24 \\ 89 \end{bmatrix}$$

$\therefore A^T A X = A^T b \Rightarrow \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 24 \\ 89 \end{bmatrix}$

$\therefore$ augmented matrix $= \begin{bmatrix} 5 & 15 & 24 \\ 15 & 55 & 89 \end{bmatrix}$ $R_1 \to \frac{R_1}{5}$

$$\sim \begin{bmatrix} 1 & 3 & 24/5 \\ 15 & 55 & 89 \end{bmatrix} \quad R_2 \to R_2 - 15R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 24/5 \\ 0 & 10 & 17 \end{bmatrix} \quad R_2 \to \frac{R_2}{10} \sim \begin{bmatrix} 1 & 3 & 24/5 \\ 0 & 1 & 17/10 \end{bmatrix}$$

$\therefore a_0 + 3b_0 = \frac{24}{5}$ and $b_0 = \frac{17}{10} = 1.7$

$\therefore a_0 + 3 \times \frac{17}{10} = \frac{24}{5}$

$\Rightarrow a_0 = \frac{24}{5} - \frac{51}{10} = \frac{3}{10} = 0.3$

$\therefore$ The best fit line is $\underline{y = 0.3 + 1.7x}$

4. Find the least squares regression line for the following data.

x	1	2	3	4	5
y	2	3	5	9	7

Ans: Let $y = a_0 + b_0 x$ be the least squares regression line.

Then a_0 and b_0 are given by the least square solution of $AX = b$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$

$X = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 7 \end{bmatrix}$

By solving its normal equations $A^T A X = A^T b$, we get X.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 26 \\ 94 \end{bmatrix}$$

$\therefore A^TAx = A^Tb \Rightarrow \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 26 \\ 94 \end{bmatrix}$

Augmented matrix $= \begin{bmatrix} 5 & 15 & 26 \\ 15 & 55 & 94 \end{bmatrix} \quad R_1 \to \frac{R_1}{5}$

$$\sim \begin{bmatrix} 1 & 3 & 26/5 \\ 15 & 55 & 94 \end{bmatrix} \quad R_2 \to R_2 - 15R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 26/5 \\ 0 & 10 & 16 \end{bmatrix} \quad R_2 \to \frac{R_2}{10}$$

$$\sim \begin{bmatrix} 1 & 3 & 26/5 \\ 0 & 1 & 16/10 \end{bmatrix}$$

$\therefore a_0 + 3b_0 = \frac{26}{5}$ and $b_0 = \frac{16}{10} = 1.6$

$a_0 + 3 \times \frac{16}{10} = \frac{26}{5} \Rightarrow a_0 = \frac{26}{5} - \frac{48}{10} = \frac{4}{10} = 0.4$

$\therefore y = \underline{\underline{0.4 + 1.6x}}$ is the least squares regression line.

5. Find the least squares regression quadratic polynomial for the following data. $(0,0)$, $(2,2)$, $(3,6)$, $(4,12)$.

Ans: Let $y = ax^2 + bx + c$ be the quadratic polynomial. By substituting given points on the quadratic polynomial we obtain the following system of linear equations.

$$a \cdot 0 + b \cdot 0 + c = 0$$
$$a \cdot 4 + b \cdot 2 + c = 2$$
$$a \cdot 9 + b \cdot 3 + c = 6$$
$$a \cdot 16 + b \cdot 4 + c = 12$$

The least square problem is $Ax=b$, where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 2 \\ 6 \\ 12 \end{bmatrix}$$

Normal equations are $A^TAX = A^Tb$

$$A^TA = \begin{bmatrix} 0 & 4 & 9 & 16 \\ 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 353 & 99 & 29 \\ 99 & 29 & 9 \\ 29 & 9 & 4 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 0 & 4 & 9 & 16 \\ 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 254 \\ 70 \\ 20 \end{bmatrix}$$

$$\therefore A^TAX = A^Tb \Rightarrow \begin{bmatrix} 353 & 99 & 29 \\ 99 & 29 & 9 \\ 29 & 9 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 254 \\ 70 \\ 20 \end{bmatrix}$$

Augmented matrix $= \begin{bmatrix} 353 & 99 & 29 & 254 \\ 99 & 29 & 9 & 70 \\ 29 & 9 & 4 & 20 \end{bmatrix}$

$R_1 \to \dfrac{R_1}{353} \sim \begin{bmatrix} 1 & 99/353 & 29/353 & 254/353 \\ 99 & 29 & 9 & 70 \\ 29 & 9 & 4 & 20 \end{bmatrix}$

$R_2 \to R_2 - 99R_1$
$R_3 \to R_3 - 29R_1$ $\sim \begin{bmatrix} 1 & 99/353 & 29/353 & 254/353 \\ 0 & 436/353 & 306/353 & -436/353 \\ 0 & 306/353 & 571/353 & -306/353 \end{bmatrix}$

$$R_2 \to R_2 \times \frac{353}{436} \sim \begin{bmatrix} 1 & 99/353 & 29/353 & 254/353 \\ 0 & 1 & 306/436 & -1 \\ 0 & 306/353 & 571/353 & -306/353 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{306}{353} R_2 \sim \begin{bmatrix} 1 & 99/353 & 29/353 & 254/353 \\ 0 & 1 & 306/436 & -1 \\ 0 & 0 & 1.009 & 0 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{1.009} \sim \begin{bmatrix} 1 & 99/353 & 29/353 & 254/353 \\ 0 & 1 & 306/436 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ $a + \frac{99}{353} b + \frac{29}{353} c = \frac{254}{353}$

$b + \frac{306}{436} c = -1$

$c = 0$

$\therefore$ $b = -1$

$a + \frac{99}{353} \times -1 + 0 = \frac{254}{353}$

$\Rightarrow a = \frac{254}{353} + \frac{99}{353} = \frac{353}{353} = 1$

$\therefore$ $a = 1,\ b = -1,\ c = 0$

$\therefore$ The quadratic polynomial is $\underline{\underline{y = x^2 - x}}$