THE INCLUSION–EXCLUSION PRINCIPLE

Theorem: For any finite sets A, B, C we have $n(A \cup B \cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B \cap C)+n(A\cap B \cap C)$ That is, we "include" n(A), n(B), n(C), we "exclude" $n(A\cap B)$, $n(A\cap C)$, $n(B \cap C)$, and finally "include" $n(A \cap B \cap C)$.

EXAMPLE 5.11 Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian, given the following data:

65 study French,	20 study French and German,	
45 study German,	25 study French and Russian,	8 study all three languages.
42 study Russian,	15 study German and Russian,	

We want to find $n(F \cup G \cup R)$ where F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the Inclusion–Exclusion Principle,

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Namely, 100 students study at least one of the three languages.

PIGEONHOLE PRINCIPLE

5.19. Find the minimum number n of integers to be selected from $S = \{1, 2, \ldots, 9\}$ so that: (a) The sum of two of the n integers is even. (b) The difference of two of the n integers is 5.

(a) The sum of two even integers or of two odd integers is even. Consider the subsets $\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8\}$ of S as pigeonholes. Hence $n = 3$.

(b) Consider the five subsets $\{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5\}$ of S as pigeonholes. Then $n = 6$ will guarantee that two integers will belong to one of the subsets and their difference will be 5.

5.20. Find the minimum number of students needed to guarantee that five of them belong to the same class (Freshman, Sophomore, Junior, Senior).

Here the $n = 4$ classes are the pigeonholes and $k + 1 = 5$ so $k = 4$. Thus among any $kn + 1 = 17$ students (pigeons), five of them belong to the same class.

5.21. Let L be a list (not necessarily in alphabetical order) of the 26 letters in the English alphabet (which consists of 5 vowels, A, E, I, O, U, and 21 consonants).

(a) Show that L has a sublist consisting of four or more consecutive consonants.

(b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

(a) The five letters partition L into $n = 6$ sublists (pigeonholes) of consecutive consonants. Here $k + 1 = 4$ and so $k = 3$. Hence $nk + 1 = 6(3) + 1 = 19 < 21$. Hence some sublist has at least four consecutive consonants.

(b) Since L begins with a vowel, the remainder of the vowels partition L into $n = 5$ sublists. Here $k + 1 = 5$ and so $k = 4$. Hence $kn + 1 = 21$. Thus some sublist has at least five consecutive consonants.

INCLUSION–EXCLUSION PRINCIPLE

5.22. There are 22 female students and 18 male students in a classroom. Find the total number t of students.

The sets of male and female students are disjoint; hence $t = 22 + 18 = 40$.

5.23. Suppose among 32 people who save paper or bottles (or both) for recycling, there are 30 who save paper and 14 who save bottles. Find the number m of people who:

(a) save both; (b) save only paper; (c) save only bottles.

Let P and B denote the sets of people saving paper and bottles, respectively. Then:

$$\begin{aligned} &(a) \quad m = n(P \cap B) = n(P) + n(B) - n(P \cup B) = 30 + 14 - 32 = 12 \\ &(b) \quad m = n(P \setminus B) = n(P) - n(P \cap B) = 30 - 12 = 18 \\ &(c) \quad m = n(B \setminus P) = n(B) - n(P \cap B) = 14 - 12 = 2 \end{aligned}$$

RECURRENCE RELATIONS

EXAMPLE 6.7 Consider the following sequence which begins with the number 3 and for which each of the following terms is found by multiplying the previous term by 2:

$$3, \quad 6, \quad 12, \quad 24, \quad 48, \quad \ldots$$

It can be defined recursively by:

$$a_0 = 3, \quad a_k = 2a_{k-1} \text{ for } k \geq 1 \quad \text{or} \quad a_0 = 3, \quad a_{k+1} = 2a_k \quad \text{for } k \geq 0$$

The second definition may be obtained from the first by setting $k = k + 1$. Clearly, the formula $a_n = 3(2^n)$ gives us the nth term of the sequence without calculating any previous term.

6.11. Consider the second-order homogeneous recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$, $a_1 = 7$,

(a) Find the next three terms of the sequence.

(b) Find the general solution.

(c) Find the unique solution with the given initial conditions.

(a) Each term is the sum of the preceding term plus twice its second preceding term. Thus:

$$a_2 = 7 + 2(2) = 11, \quad a_3 = 11 + 2(7) = 25, \quad a_4 = 25 + 2(11) = 46$$

(b) First we find the characteristic polynomial $\Delta(t)$ and its roots:

$$\Delta(x) = x^2 - x - 2 = (x - 2)(x + 1); \quad \text{roots } r_1 = 2, r_2 = -1$$

Since the roots are distinct, we use Theorem 6.8 to obtain the general solution:

$$a_n = c_1(2^n) + c_2(-1)^n$$

(c) The unique solution is obtained by finding c_1 and c_2 using the initial conditions:

$$\begin{aligned} &\text{For } n = 0, a_0 = 2, \text{ we get:} \quad c_1(2^0) + c_2(-1)^0 = 2 \quad \text{or} \quad c_1 + c_2 = 2 \\ &\text{For } n = 1, a_1 = 7, \text{ we get:} \quad c_1(2^1) + c_2(-1)^1 = 7 \quad \text{or} \quad 2c_1 - c_2 = 7 \end{aligned}$$

Solving the two equations for c_1 and c_2 yields $c_1 = 3$ and $c_2 = 1$. The unique solution follows:

$$a_n = 3(2^n) - (-1)^n$$

Algebraic Systems

OPERATIONS AND SEMIGROUPS

B.1. Consider the set **Q** of rational numbers, and let $*$ be the operation on **Q** defined by

$$a * b = a + b - ab$$

(*a*) Find: (i) $3 * 4$; (ii) $2 * (-5)$; (iii) $7 * (1/2)$.

(*b*) Is $(\mathbf{Q}, *)$ a semigroup? Is it commutative?

(*c*) Find the identity element for $*$.

(*d*) Do any of the elements in **Q** have an inverse? What is it?

(*a*) (i) $3*4 = 3 + 4 - 3(4) = 3 + 4 - 12 = -5$

(ii) $2 * (-5) = 2 + (-5) + 2(-5) = 2 - 5 + 10 = 7$

(iii) $7 * (1/2) = 7 + (1/2) - 7(1/2) = 4$

(*b*) We have:

$$\begin{aligned}(a * b) * c &= (a + b - ab) * c = (a + b - ab) + c - (a + b - ab)c \\ &= a + b - ab + c - ac - bc + abc = a + b + c - ab - ac - bc + abc \\ a * (b * c) &= a * (b + c - bc) = a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc\end{aligned}$$

Hence $*$ is associative and $(\mathbf{Q}, *)$ is a semigroup. Also

$$a * b = a + b - ab = b + a - ba = b * a$$

Hence $(\mathbf{Q}, *)$ is a commutative semigroup.

(*c*) An element e is an identity element if $a * e = a$ for every $a \in \mathbf{Q}$. Compute as follows:

$$a * e = a, \quad a + e - ae = a, \quad e - ea = 0, \quad e(1 - a) = 0, \quad e = 0$$

Accordingly, 0 is the identity element.

(*d*) In order for a to have an inverse x, we must have $a * x = 0$ since 0 is the identity element by Part (*c*). Compute as follows:

$$a * x = 0, \quad a + x - ax = 0, \quad a = ax - x, \quad a = x(a - l), \quad x = a/(a - l)$$

Thus if $a \neq 1$, then a has an inverse and it is $a/(a - 1)$.

B.2. Let S be a semigroup with identity e, and let b and b' be inverses of a. Show that $b = b'$, that is, that inverses are unique if they exist.

We have:

$$b * (a * b') = b * e = b \quad \text{and} \quad (b * a) * b' = e * b' = b'$$

Since S is associative, $(b * a) * b' = b* (a * b')$; hence $b = b'$.

B.3. Let $S = \mathbf{N} \times \mathbf{N}$. Let $*$ be the operation on S defined by $(a, b) * (a', b') = (aa', bb')$.

(*a*) Show that $*$ is associative. (Hence S is a semigroup.)

(*b*) Define $f: (S, *) \to (\mathbf{Q}, \times)$ by $f(a, b) = a/b$. Show that f is a homomorphism.

(*c*) Find the congruence relation $\sim$ in S determined by the homomorphism f, that is, where $x \sim y$ if $f(x) = f(y)$. (See Theorem B.4.)

(*d*) Describe $S/\sim$. Does $S/\sim$ have an identity element? Does it have inverses?

Suppose $x = (a, b)$, $y = (c, d)$, $z = (e, f)$.

(*a*) We have

$$(xy)z = (ac, bd) * (e, f) = [(ac)e, (bd)f]$$
$$x(yz) = (a, b) * (ce, df) = [a(ce), b(df)]$$

Since a, b, c, d, e, f, are positive integers, $(ac)e = a(ce)$ and $(bd)f = b(df)$. Thus $(xy)z = x(yz)$ and hence $*$ is associative. That is, $(S, *)$ is a semigroup.

(*b*) f is a homomorphism since

$$f(x * y) = f(ac, bd) = (ac)/(bd) = (a/b)(c/d) = f(x)f(y)$$

(*c*) Suppose $f(x) = f(y)$. Then $a/b = c/d$ and hence $ad = bc$. Thus f determines the congruence relation $\sim$ on S defined by $(a, b) \sim (c, d)$ if $ad = bc$.

(*d*) The image of f is $\mathbf{Q}^+$, the set of positive rational numbers. By Theorem B.3, $S/\sim$ is isomorphic to $\mathbf{Q}^+$. Thus $S/\sim$ does have an identity element, and every element has an inverse.

GROUPS

B.6. Consider the group $G = \{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.

(a) Find the multiplication table of G.
(b) Find $2^{-1}, 3^{-1}, 6^{-1}$.
(c) Find the orders and subgroups generated by 2 and 3.
(d) Is G cyclic?

(a) To find $a * b$ in G, find the remainder when the product ab is divided by 7.
For example, $5 \cdot 6 = 30$ which yields a remainder of 2 when divided by 7; hence $5 * 6 = 2$ in G. The multiplication table of G appears in Fig. B-6(a).

(b) Note first that 1 is the identity element of G. Recall that a^{-1} is that element of G such that $aa^{-1} = 1$. Hence $2^{-1} = 4$, $3^{-1} = 5$ and $6^{-1} = 6$.

(c) We have $2^1 = 2$, $2^2 = 4$, but $2^3 = 1$. Hence $|2| = 3$ and $gp(2) = \{1, 2, 4\}$. We have $3^1 = 3$, $3^2 = 2$, $3^3 = 6$, $3^4 = 4$, $3^5 = 5$, $3^6 = 1$. Hence $|3| = 6$ and $gp(3) = G$.

(d) G is cyclic since $G = gp(3)$.

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(a)

*	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

(b)

B.7. Let G be a reduced residue system modulo 15, say, $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$ (the set of integers between 1 and 15 which are coprime to 15). Then G is a group under multiplication modulo 15.

(*a*) Find the multiplication table of G.

(*b*) Find $2^{-1}, 7^{-1}, 11^{-1}$.

(*c*) Find the orders and subgroups generated by 2, 7, and 11.

(*d*) Is G cyclic?

(*a*) To find $a * b$ in G, find the remainder when the product ab is divided by 15. The multiplication table appears in Fig. B-6(*b*).

(*b*) The integers r and s are inverses if $r * s = 1$. Hence: $2^{-1} = 8$, $7^{-1} = 13$, $11^{-1} = 11$.

(*c*) We have $2^2 = 4$, $2^3 = 8$, $2^4 = 1$. Hence $|2| = 4$ and $\text{gp}(2) = \{1, 2, 4, 8\}$. Also, $7^2 = 4$, $7^3 = 4 * 7 = 13$, $7^4 = 13 * 7 = 1$. Hence $|7| = 4$ and $\text{gp}(7) = \{1, 4, 7, 13\}$. Lastly, $11^2 = 1$. Hence $|11| = 2$ and $\text{gp}(11) = \{1, 11\}$.

(*d*) No, since no element generates G.