22/03/25
Saturday

MODULE : 4

Laplace transforms

If $F(t)$ is a function defined for all $t \geq 0$, then its laplace transforms is given by :

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt$$

Note :

$L[F(t)]$ is also denoted as $F(s)$.

Basic laplace transforms

① $L[1] = 1/s$

② $L[e^{at}] = 1/s-a$

③ $L[e^{-at}] = \frac{1}{s+a}$

④ $L[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$

⑤ $L[\cos \omega t] = \frac{s}{s^2+\omega^2}$

⑥ $L[\sin h\omega t] = \frac{\omega}{s^2-\omega^2}$

⑦ $L[\cos h\omega t] = \frac{s}{s^2-\omega^2}$

⑧ $L[t^n] = \frac{n!}{s^{n+1}}$

⑨ $L[t] = \frac{1!}{s^2} = \frac{1}{s^2}$

⑩ $L[t^2] = \frac{2!}{s^3} = \frac{2}{s^3}$

Q1. Find the laplace transform of the following

① $L[\cos 2\pi t]$ $\qquad L[\cos\omega t] = \frac{s}{s^2+\omega^2}$

$$L[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

here $\omega = 2\pi$

$$\therefore L[\cos 2\pi t] = \frac{s}{s^2+(2\pi)^2}$$

$$= \frac{s}{s^2+4\pi^2}$$

② $L[2t+2] = L[2t] + L[2]$

$$= 2L[t] + 2L[1]$$

$$= 2 \cdot \frac{1}{s^2} + 2 \cdot \frac{1}{s}$$

③ $L[\underbrace{(2t+1)^2}_{(a+b)^2}] = L[4t^2+4t+1]$

$$= L[4t^2] + L[4t] + L[1]$$

$$= 4L[t^2] + 4L[t] + L[1]$$

$$= 4 \cdot \frac{2}{s^3} + 4 \cdot \frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{8}{s^3} + \frac{4}{s^2} + \frac{1}{s}$$

(4) $L[(t-1)^3] =$

$(a-b)^3 = a^3 - 3ab^2 + 3a^2$

$= L[t^3 - 1^3 - 3t^2 + 3t]$

$= L[t^3] - L[1] - 3L[t^2] + 3L[t]$

$= L[t^3] - 3L[t^2] + 3L[t] - L[1]$

$L[t^n] = \frac{n!}{s^{n+1}}$

i.e, $L[t^3] = \frac{3!}{s^4}$

$L[t^2] = \frac{2!}{s^3}$

$L[t] = \frac{1}{s^2}$

$= \frac{3!}{s^4} - \frac{3 \cdot 2}{s^3} + 3\frac{1}{s^2} - \frac{1}{s}$

(5) $L[(a-bt)^2]$

$= L[a^2 - 2abt + bt^2]$

$= a^2 L[1] - 2ab L[t] + b L[t^2]$

$= a^2 \frac{1}{s} - 2ab \frac{1}{s^2} + b\frac{2}{s^3}$

6) Find the laplace transform of $(\sin t + \cos t)^2$?

$L[(\sin t + \cos t)^2] = L[\sin^2 t + 2 \sin t \cos t + \cos^2 t]$

$= L[\sin^2 t + \cos^2 t + 2 \sin t \cos t]$

$= L[1] + L[\sin 2t]$

here $\omega = 2$

$= \frac{1}{s} + \frac{2}{s^2+4}$ //

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Note:

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\cos^2 2\theta = \dfrac{1+\cos 4\theta}{2}$

$\sin^2 2\theta = \dfrac{1-\cos 4\theta}{2}$

1. Find $L[\cos^2 \omega t]$.

$= L\left[\dfrac{1+\cos 2\omega t}{2}\right]$

$= \frac{1}{2} L[1+\cos 2\omega t]$

$= \frac{1}{2}\left[L(1) + L(\cos 2\omega t)\right]$

$= \frac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+4\omega^2}\right]$

2. Find $L[\sin^2 2t]$

$= L\left[\dfrac{1-\cos 4t}{2}\right]$

$= 1/2\, L[1-\cos 4t]$

$= 1/2\left(L[1] - L[\cos 4t]\right)$

$= 1/s\left[\dfrac{1}{s} - \dfrac{s}{s^2+16}\right]$

3. Find $L[\cos(\omega t+\theta)]$ $\boxed{\cos(a+b)}$

$= L[\cos\omega t\cos\theta - \sin\omega t\sin\theta]$

$= \cos\theta\, L[\cos\omega t] - \sin\theta\, L[\sin\omega t]$

$= \cos\theta\, \dfrac{s}{s^2+\omega^2} - \sin\theta\, \dfrac{\omega}{s^2+\omega^2}$

4 Find $L[\sin 3t \cos 2t]$

$$= L\left[\tfrac{1}{2}\left[\sin(3t+2t) + \sin(3t-2t)\right]\right]$$

$$= \tfrac{1}{2}\left[L\left[\sin 5t + \sin t\right]\right]$$

$$= \tfrac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$2 \sin A \cos B$
$= \sin(A+B)$
$\sin(A-$

Properties of Laplace Transforms:

* I^{st} shifting theorm

IF $L[F(t)] = F(s)$ then

$$L[e^{at} F(t)] = F(s-a)$$

1\. Find $L[e^{3t} t^2]$

here $F(t) = t^2$

$$L[F(t)] = L[t^2] = \frac{2}{s^3}$$

so $F(s) = \dfrac{2}{s^3}$

here $a = 3$

so $F(s-a) = \dfrac{2}{(s-3)^3}$

$$\therefore L[e^{3t} t^2] = \frac{2}{(s-3)^3}$$

2\. $L[e^{2t} t^2]$

$F(t) = t^2$

$$L[f(t)] = \frac{2}{s^3} = F(s)$$

$$a = 2$$

$$F(s-a) = \frac{2}{(s-2)^3}$$

$$\therefore L[e^{2t} t^2] = \frac{2}{(s-2)^3}$$

Q. find Laplace transform of following

(1) $e^{-3t} \cos 2t$ $\quad -3t = a$

$$f(t) = \cos 2t$$

$$L[F(t)] = \frac{s}{s^2+4}$$

here $a = -3$

so, $L[e^{-3t} \cos 2t] = \dfrac{s+3}{(s+3)^2+4}$

(2) $e^{2t} \cos 3t$

$$f(t) = \cos 3t$$

$$L[f(t)] = \frac{s}{s^2+9}$$

here $a = 2$

$$L[e^{2t} \cos 3t] = \frac{s-2}{(s-2)^2+9}$$

(3) $t^3 e^{-2t}$

$$f(t) = t^3$$

$$L[f(t)] = \frac{3!}{s^4}$$

here $a = -2$ $\quad L[t^3 e^{-2t}] = \dfrac{3!}{(s+2)^4}$ //

Inverse laplace transform

Let $L[f(t)] = F(s)$ be the laplace transform then the inverse laplace transform is denoted by $f(t) = L^{-1}[F(s)]$

$F(s)$	$f(t) = L^{-1}[F(s)]$
0	0
$\frac{1}{s^2}$	t
$1/s$	1
$1/s-a$	e^{at}
$1/s+a$	e^{-at}
$1/s^2+\omega^2$	$\frac{1}{\omega}\sin\omega t$
$\frac{s}{s^2+\omega^2}$	$\cos\omega t$
$\frac{1}{s^2-\omega^2}$	$\frac{1}{\omega}\sinh\omega t$
$\frac{s}{s^2-\omega^2}$	$\cosh\omega t$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$

1. $L^{-1}\left[\frac{5s+1}{s^2-25}\right] = L^{-1}\left[\frac{5s}{\underbrace{s^2-25}_{5^2}}\right] + L^{-1}\left[\frac{1}{s^2-25}\right]$

$= 5L^{-1}\left[\frac{s}{s^2-5^2}\right] + L^{-1}\left[\frac{1}{s^2-5^2}\right]$

$= 5\cosh 5t + \frac{1}{5}\sinh 5\omega t$

2. $L^{-1}\left[\frac{2s+14}{s^2+196}\right] = L^{-1}\left[\frac{2s}{s^2+196}\right] + L^{-1}\left[\frac{14}{s^2+196}\right]$

$\sqrt{196} = 14$

$= 2L^{-1}\left[\frac{s}{s^2+14^2}\right] + 14L^{-1}\left[\frac{1}{s^2+14^2}\right]$

$= 2\cos 14t + 14\frac{1}{13}\sin 14t$

3. $L^{-1}\left[\frac{2}{s^4} - \frac{48}{s^6}\right]$

$n+1 = 4$
$n = 3$
$n+1 = 6$
$n = 5$

$2L^{-1}\left[\frac{1}{s^4}\right] - 48L^{-1}\left[\frac{1}{s^6}\right]$

$2 \cdot \frac{t^3}{3!} - 48\frac{t^5}{5!}$

$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

$\frac{\cancel{2}t^3}{1\times\cancel{2}\times 3} - \frac{\overset{2}{\cancel{4}\cancel{8}}\, t^5}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}\times 5}$

$= \frac{t^3}{3} - \frac{2t^5}{5}$

Q1) Find $L^{-1}\left[\frac{1}{(s+1)(s-2)}\right]$

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \quad \text{— (1)}$$

multiply by $(s-1)(s-2)$

$$1 = \frac{A\cancel{(s-1)}(s-2)}{\cancel{s-1}} + \frac{B(s-1)\cancel{(s-2)}}{\cancel{s-2}}$$

$$1 = A(s-2) + B(s-1) \quad \text{— (2)}$$

put $s = 2$ in equ (2)

$$1 = A(2-2) + B(2-1)$$

$$1 = 0 + B$$

$$B = 1$$

put $s = 1$ in equ (2)

$$1 = A(1-2) + B(1-1)$$

$$1 = A \cdot -1 + 0$$

$$A = -1$$

$\therefore$ equ (1) $\Rightarrow$

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}$$

$$L^{-1}\frac{1}{(s-1)(s-2)} = -L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right]$$

$$= -e^{t} + e^{2t}$$

2) $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad \text{—(1)}$$

multiply by $s(s+1)(s+2)$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1) \quad \text{—(2)}$$

put $s=-2$

$1 = 0 + 0 + C\cdot -2(-2+1)$

$1 = -2C(-1)$

$1 = 2C$

$C = \frac{1}{2}$

put $s=-1$

$1 = A(-1+1)(-1+2) + B\times -1(-1+2) + C\cdot 0$

$1 = 0 + -B(+1) + 0$

$1 = -B$

$\therefore B = -1$

put $s=0$

$1 = A(0+1)(0+2) + B\cdot 0 + C\cdot 0$

$1 = A(1)\cdot(2) + 0 + 0$

$1 = 2A$

$A = \frac{1}{2}$

equ (1) $\Rightarrow$

$$\frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{+\frac{1}{2}}{s+2}$$

$= \frac{1}{} - \frac{1}{} + \frac{1}{}$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1} [$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

Solution of ordinary differential equ[illegible] using laplace transform.

$L[y'] = sL[y] - y(0)$

$L[y''] = s^2L[y] - sy(0) - y'(0)$

1. using laplace transform solve $y''+5y'+6y = e^{-t}$, $y(0) = 0$ and $y'(0) = 1$

A: consider $y''+5y'+6y = e^{-t}$

$$L[y''] + 5L[y'] + 6L[y] = L[e^{-t}]$$

$$\Rightarrow [s^2L(y) - sy(0) - y'(0)] + 5[sL(y) - y(0)] + 6L[y] = L[e^{-t}]$$

$L[e^{-at}] =$

$a=1$

given that $y(0)=0$ & $y'(0)=1$ and

$$L[e^{-at}] = \frac{1}{s+a}$$

$$\Rightarrow [s^2L[y] - s\times 0 - 1] + 5[sL[y] - 0] + 6L[y] = \frac{1}{s+1}$$

$$s^2L[y] - 1 + 5sL[y] + 6L[y] = \frac{1}{s+1}$$

Taking $L[y]$ common $\Rightarrow$

$$\Rightarrow L[y][s^2+5s+6] - 1 = \frac{1}{s+1}$$

$$\Rightarrow L[y][s^2+5s+6] = \frac{1}{s+1} + 1$$

Lcm = s

$$\Rightarrow L[y][s^2+5s+6] = \frac{1+s+1}{s+1}$$

$$\Rightarrow L[y][s^2+5s+6] = \frac{s+2}{s+1}$$

[Factorise s^2+5s

$$\Rightarrow L[y](s+3)(s+2) = \frac{s+2}{s+1}$$

$$\Rightarrow L[y] = \frac{\cancel{s+2}}{(s+1)(s+3)\cancel{(s+2)}}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{(s+1)(s+3)}\right] \text{ —— } ①$$

consider, $\dfrac{1}{(s+1)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+3}$ —— ②

multiply by $(s+1)(s+3)$

$$1 = \frac{A\cancel{(s+1)}(s+3)}{\cancel{s+1}} + \frac{B(s+1)\cancel{(s+3)}}{\cancel{s+3}}$$

$$1 = A(s+3) + B(s+1) \text{ —— } ③$$

put $s = -3$ in equ ③

$$1 = A(-3+3) + B(-3+1)$$

$$1 = 0 + -2B$$

$$B = \frac{-1}{2}$$

put $s = -1$ in equ ③

$$1 = A(-1+3) + B(-1+1)$$

$$1 = A\,2 + 0$$

$$A = \frac{1}{2}$$

sub in equ ②

$$\frac{1}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{2(s+3)}$$

$$\therefore ① \Rightarrow y = L^{-1}\left[\frac{1}{2(s+1)} - \frac{1}{2(s+3)}\right]$$

$$y = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]_{a=1} - \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]_{a=3}$$

$$\Rightarrow \quad L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$\therefore \quad y = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

2. using laplace transform solve, $y''+2y'+y = e^{-t}$

$y(0) = 0 \quad , \quad y'(0) = 1$

$$y''+2y'+y = e^{-t}$$

$$L[y''] + 2L[y'] + L[y] = e^{-t}$$

$$[s^2L[y] - s\,y(0) - y'(0)] + 2[sL(y) - y(0)] + L[y] = \frac{1}{s+1}$$

put $y(0) = 0$ and $y'(0) = 1$

$$s^2L[y] - 1 + 2sL[y] + L[y] = \frac{1}{s+1}$$

$$L[y][s^2+2s+1] - 1 = \frac{1}{s+1}$$

$$L[y](s+1)^2 = \frac{1}{s+1} + 1$$

$$= \frac{s+1}{s+1} = \frac{s+2}{s+1}$$

$$L[y] = \frac{s+2}{(s+1)(s+1)^2} = \frac{s+2}{(s+1)^3}$$

$$y = L^{-1}\left[\frac{s+2}{(s+1)^3}\right]$$

here $a = -1$ $\qquad a = -1$

$$y = 1e^{-t}L^{-1}\left[\frac{1}{s^2}\right] + e^{-t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$y = e^{-t}t + e^{-t}\frac{t^2}{2}$$

3. Solve the initial value problem $y''+9y=10e^{-t}$

$y(0)=0 \qquad y'(0)=1$

$$y''+9y=10e^{-t}$$

$$L[y'']+9L[y]=10L[e^{-t}]$$

$$[s^2L[y]-sy(0)-y'(0)]+9L[y]=10L[e^{-t}]$$

$$[s^2L[y]-0-1]\ 9L[y]=10\frac{1}{s+1}$$

$$L[y]\ [s^2+9]-1=\frac{10}{s+1}$$

$$L[y]\ [s^2+9]=\frac{s+11}{s+1}$$

$$[y]=L^{-1}\left[\frac{s+11}{(s+1)(s^2+9)}\right] \quad \text{———} \ 1$$

$$\frac{11+s}{(s+1)(s^2+9)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+9)} \quad \text{———} \ 2$$

$$11+s=A(s^2+9)+(Bs+C)(s+1) \quad \text{———} \ 3$$

put $s=-1$

$10=10A+0$

$\therefore A=1$

equating the coefficients of s^2 on both sides in equ 3 $\Rightarrow$

$0=A+B$

$0=1+B$

$\therefore B=-1$

equating the coefficients of s on both sides in equ ② $\Rightarrow$

$1 = B + C$

$1 - -1 + C$

$C = 2$

$\therefore$ equ ② $\Rightarrow$

$$\frac{11+s}{(s+1)(s^2+9)} = \frac{1}{(s+1)} + \frac{-s+2}{s^2+9}$$

so equ ① $\Rightarrow$

$$y = L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{-s+2}{s^2+9}\right]$$

$$y = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s}{s^2+3^2}\right] + 2L^{-1}\left[\frac{1}{s^2+}\right.$$

$$y = e^{-t} - \cos 3t + \frac{2}{3}\sin 3t$$