Module 4 : Syllubus Discription (9Hrs)

Taylor Series expansion (without proof, assuming the possibility of power Series expansion in appropriate domain.), Maclaurin Series representation, Fourier Series, Euler formulas, Convergence of fourier Series (Dirichlet's conditions). Fourier Series of 2π periodic functions, Fourier Series of $2l$ periodic functions, Half range sin series expansion, Half range cosine series expansion.

Taylor Series Expansion of f(x)

If f(x) has derivatives of all orders at the point x_0, then the Series

$$f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \cdot f^n(x_0)$$

is called the Taylor series expansion of f(x) about $x=x_0$. ie, the Taylor Series expansion of

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \cdot f^n(x_0). \text{ at } x=x_0}$$

Maclaurin Series Expansion of f(x)

The Taylor Series about $x=0$ is Called the Maclaurin Series of f(x).

ie, Maclaurin series of

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-0)^n}{n!} f^n(0) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^n(0)$$

$$= f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Que 1:- Find the Taylor Series Expansion of $f(x) = \log x$ about the point $x=1$.

Ans:- Taylor series expansion of

$$f(x) = f(x_0) + \frac{x-x_0}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

Here $x_0 = 1$ and

$$f(x) = \log x. \qquad f(1) = \log 1 = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \qquad f'(1) = \frac{1}{1} = 1$$

$$f''(x) = \frac{-1}{x^2} \quad (-1x^{-2}) \qquad f''(1) = \frac{-1}{1} = -1$$

$$f'''(x) = \frac{2}{x^3} \quad (2x^{-3}) \qquad f'''(1) = \frac{2}{1} = 2.$$

$$f^{iv}(x) = \frac{-6}{x^4} \quad (-6x^{-4}) \qquad f^{iv}(x) = \frac{-6}{1} = -6.$$

$$\therefore f(x) = 0 + \frac{x-1}{1!} \times 1 + \frac{(x-1)^2}{2!} \times (-1) + \frac{(x-1)^3}{3!} \times (2) + \frac{(x-1)^4}{4!} \times (-6) + \cdots$$

$$\Rightarrow f(x) = (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!} + \cdots$$

Que 2:- Expand e^x in ~~powers~~ terms of $(x-1)$ using Taylor series.

Ans:- Here $x_0 = 1$

$$f(x) = e^x \qquad f(1) = e^1 = e$$

$$f'(x) = e^x \qquad f'(1) = e^1 = e$$

$$f'''(x) = e^x \qquad f'''(1) = e^1 = e. \quad \cdots$$

$$\therefore f(x) = f(1) + \frac{(x-1)}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots$$

$$= e + \frac{x-1}{1!} e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \cdots$$

$$= e\left[1 + \frac{x-1}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

Que 3:- Find the Maclaurin Series expansion of $\cos x$ [or Taylor series expansion of $\cos x$ at $x=0$].

Ans:- Here $x_0=0$ and $f(x)=\cos x$.

$$f(x) = f(x_0) + \frac{x-x_0}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

$$\Rightarrow f(x) = f(0) + \frac{x-0}{1!}f'(0) + \frac{(x-0)^2}{2!}f''(0) + \cdots$$

$$\Rightarrow f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$f(x) = \cos x \implies f(0) = \cos 0 = 1$

$f'(x) = -\sin x \implies f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \implies f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \implies f'''(0) = +\sin 0 = 0$

$f^{IV}(x) = \cos x \implies f^{IV}(0) = \cos 0 = 1$

$$\therefore f(x) = 1 + \frac{x}{1!}\times 0 + \frac{x^2}{2!}\times(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Que 4:- Find the Maclaurin series expansion of $\sin x$

Ans:- Here $x_0=0$ and $f(x)=\sin x$.

Expansion,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$f(x) = \sin x \implies f(0) = \sin 0 = 0$

$f'(x) = \cos x \implies f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \implies f''(0) = -\sin 0 = 0.$

$f'''(x) = -\cos x \implies f'''(0) = -\cos 0 = -1$

$f^{IV}(x) = \sin x \implies f^{IV}(0) = \sin 0 = 0$

$f^{V}(x) = \cos x \implies f^{V}(0) = \cos 0 = 1$

$$\therefore f(x) = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times (-1) + \frac{x^4}{4!} \times 0 + \frac{x^5}{5!} \times 1 + \cdots$$

$$\Rightarrow f(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Que 5: Find the Maclaurin series expansion of $x \sin x$.

Ans: Here $x_0 = 0$ and $f(x) = x \sin x$.

$$f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \cdots$$

$$\Rightarrow f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$f(x) = x \sin x \Longrightarrow f(0) = 0$$

$$f'(x) = [\sin x + x \cos x] \Longrightarrow f'(0) = 0$$

$$f''(x) = \cos x + \cos x + x\,{}^{-}\sin x \Longrightarrow f''(0) = 2$$

$$f'''(x) = {}^{-}\sin x\ {}^{-}\sin x - (\sin x + x \cos x) \Rightarrow f'''(0) = 0.$$

$$f^{iv}(x) = {}^{-}\cos x - \cos x - \cos x - (\cos x - x \sin x)$$

$$\Longrightarrow f^{iv}(0) = -4$$

$$\therefore f(x) = 0 + 0 + \frac{x^2}{2!} \times 2 + 0 + \frac{x^4}{4!} \times -4 + \cdots$$

$$\Rightarrow f(x) = \frac{2x^2}{2!} - \frac{4x^4}{4!} + \cdots$$

Fourier Series

Periodic functions:- A function f(x) is said to be periodic if the value of f(x) repeates exactly after a regular interval of time.

ie, $f(x)=f(x+T)$ where T is a +ve constant.

The least value of T is called the period of f(x).

Eg:- $f(x)=\sin x=\sin(x+2\pi)=\sin(x+4\pi)=\sin(x+6\pi)\ldots$

∴ The function has periods $2\pi, 4\pi, 6\pi \ldots$ However 2π is the least value and is the period of f(x).

Similarly, cos x, sec x and cosec x are periodic functions with period 2π and tan x, cot x are periodic functions with period π.

Fourier Series:- If f(x) is a periodic function with period $2l$ [ie, f(x) in the interval $(c, c+2l)$], then it can be represented by an infinite series called Fourier series as

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b\sin\left(\frac{n\pi x}{l}\right)\right]}$$

where a_0, a_n and b_n are called Fourier Coefficients.

and are defined by the Euler formula:-

$$a_0=\frac{1}{l}\int_{c}^{c+2l}f(x)\,dx$$

$$a_n=\frac{1}{l}\int_{c}^{c+2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n=\frac{1}{l}\int_{c}^{c+2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Results:- ① Odd and Even functions:-

A function f(x) is said to be odd if $f(x)=-f(x)$

Eg:- ① $f(x) = x$.

$f(-x) = -x = -f(x)$

② $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$.

③ $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$.

$\boxed{\sin x \text{ is an odd function}}$

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$.

Eg:- ① $f(x) = x^2$.

$f(-x) = (-x)^2 = x^2 = f(x)$

② $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$

$\boxed{\cos x \text{ is an even function}}$

Result ②:- Even × Even = even.

odd × odd = even

even × odd = odd.

Result ③:- If $f(x)$ is even function on $[-a, a]$, then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

If $f(x)$ is odd in $[-a, a]$, then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0.}$$

Result 4:- $\cos(n\pi) = (-1)^n$

$\sin(n\pi) = 0$.

Que 1:- Find the fourier series expansion of $f(x) = x$ in the interval $[0, 2\pi]$

Ans:- The fourier series expansion of $f(x)$ is defined by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where $a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Here $[c, c+2l] = [0, 2\pi]$

$$\Rightarrow 2l = 2\pi \Rightarrow l = \pi$$

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^2}{2} - \frac{0}{2}\right] = \frac{1}{\pi}\left[\frac{4\pi^2}{2} - 0\right]$$

$$= \frac{1}{\pi} \times \frac{4\pi^2}{2} = \underline{\underline{2\pi}}$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cos(nx)\,dx.$$

$$= \frac{1}{\pi}\left[x \cdot \frac{\sin(nx)}{n} - 1 \cdot \frac{-\cos(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x \cdot \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{2\pi \cdot \sin(2\pi n)}{n} + \frac{\cos(2\pi n)}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos(2\pi n)}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi}\left[\frac{(-1)^{2n}}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = \underline{\underline{0}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \cdot \sin\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_0^{2\pi} x \cdot \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[x \cdot \frac{-\cos(nx)}{n} - 1 \cdot \frac{-\sin(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-2\pi\cos(2n\pi)}{n} + \frac{\sin(2n\pi)}{n^2}\right] - \left[0+0\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi(-1)^{2n}}{n} + 0\right] = \frac{1}{\pi}\left[\frac{-2\pi}{n}\right] = -2/n.$$

$$\therefore f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left[0 + \frac{-2}{n}\cdot\sin\left(\frac{n\pi x}{\pi}\right)\right]$$

$$\Rightarrow f(x) = \pi + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin(nx)$$

2. Find the fourier ________________. series of the periodic function $f(x) = x\sin x$, in the interval $0<x<2\pi$

Ans:- Here period $2l = 2\pi \rightarrow l = \pi$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\pi}\right) + b_n\sin\left(\frac{n\pi x}{\pi}\right)\right]$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$

$$= \frac{1}{\pi}\left[x \cdot -\cos x - 1 \cdot -\sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x\cos x + \sin x\right]_0^{2\pi} = \frac{1}{\pi}\left[\left(-2\pi(-1)^{2} + 0\right) - (0+0)\right]$$

$$= \frac{1}{\pi} \times -2\pi = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \cos(nx) dx$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left[\frac{1}{2}\left(\sin\left(x+nx\right) + \sin\left(x-nx\right)\right)\right] dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\left[\sin(1+n)x + \sin(1-n)x\right] dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot \sin(1+n)x \, dx + \frac{1}{2\pi}\int_0^{2\pi} x \cdot \sin(1-n)x \, dx.$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{-\cos(1+n)x}{1+n} - 1 \cdot \frac{-\sin(1+n)x}{(1+n)^2}\right]_0^{2\pi} + \frac{1}{2\pi}\left[x \cdot \frac{-\cos(1-n)x}{1-n} - 1 \cdot \frac{-\sin(1-n)x}{(1-n)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[\cancel{\frac{-x\cos(1+n)2\pi}{1+n}} - 0\right] \cancel{- \left[0 - 0\right]}\right\} - \cancel{\frac{1}{2\pi}}\left[\cancel{\frac{-x\cos(1-n)2\pi}{1-n}}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{-2\pi\cos(1+n)2\pi}{1+n} - 0\right) - \left(0 - 0\right)\right] + \frac{1}{2\pi}\left[\left(-\frac{2\pi\cos(1-n)2\pi}{1-n}\right)\right]$$

$$= \frac{1}{2\pi} \times \left[-\frac{2\pi\cos(1+n)2\pi}{1+n}\right] + \frac{1}{2\pi}\left[-\frac{2\pi\cos(1-n)2\pi}{1-n}\right]$$

$$= \frac{1}{2\pi} \times \frac{-2\pi\,(-1)^{2(1+n)}}{1+n} + \frac{1}{2\pi} \times \frac{-2\pi\,(-1)^{2(1-n)}}{1-n}.$$

$$= \frac{-1 \times 1}{n+1} - \frac{1}{1-n} = \frac{-1}{1+n} - \frac{1}{1-n}$$

$$= \frac{-1 + n - (1+n)}{(1+n)(1-n)} = \frac{-2}{1^2 - n^2} = \underline{\underline{\frac{-2}{1-n^2}}} = \underline{\underline{\frac{2}{n^2-1}}} \quad n \neq 1$$

when $n=1$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cdot \frac{1}{2}\sin 2x \; dx \qquad \sin 2x = 2\sin x \cos x$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx = \frac{1}{2\pi}\left[x \cdot \frac{-\cos 2x}{2} - 1 \cdot \frac{-\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-2\pi \cos 4\pi}{2} + 0\right) - (0 + 0)\right]$$

$$= \frac{1}{2\pi} \times \frac{-2\pi \cdot (-1)^4}{2} = -1/2 .$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \sin\left(\frac{n\pi x}{\pi} dx\right)$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cdot \sin x \cdot \sin(nx)\, dx .$$

$$\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left[\frac{1}{2}\left(\cos(x-nx)\right\} - \cos(x+nx)\right] dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(1-n)x - \cos(1+n)x\right] dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(1-n)x \, dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(1+n)x \, dx$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{\sin(1-n)x}{1-n} - 1 \cdot \frac{-\cos(1-n)x}{(1-n)^2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x \cdot \frac{\sin(1+n)x}{1+n} - 1 \cdot \frac{-\cos(1+n)x}{(1+n)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(0 + \frac{\cos(1-n)2\pi}{(1-n)^2}\right) - \left(0 + \frac{1}{(1-n)^2}\right)\right] - \frac{1}{2\pi}\left[\left(0 + \frac{\cos(1+n)2\pi}{(1+n)^2}\right) - \left(0 + \frac{1}{(1+n)^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{2(1-n)}}{(1-n)^2} - \frac{1}{(1-n)^2}\right] - \frac{1}{2\pi}\left[\frac{(-1)^{2(1+n)}}{(1+n)^2} - \frac{1}{(1+n)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(1-n)^2} - \frac{1}{(1-n)^2}\right] + \frac{1}{2\pi}\left[\frac{1}{(1+n)^2} - \frac{1}{(1+n)^2}\right]$$

$$= \frac{1}{2\pi}\left[0\right] = \underline{\underline{0}} . \qquad n \neq 1$$

When $n=1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cdot \sin x\,dx$.

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cdot\frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos 2x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\cdot\frac{\sin 2x}{2} - 1\cdot\frac{-\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2-0}{2}\right] - \frac{1}{2}\left[\left(0+\frac{\cos 4\pi}{4}\right)-\left(0+\frac{\cos 0}{4}\right)\right]$$

$$= \frac{1}{2\pi}\times 2\pi^2 - \frac{1}{2}\left[\frac{1}{4}-\frac{1}{4}\right]$$

$$= \underline{\underline{\pi}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\pi}\right) + b_n\sin\left(\frac{n\pi x}{\pi}\right)\right]$$

$$\Rightarrow f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + \sum_{n=2}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$\Rightarrow f(x) = \frac{-2}{2} + -\frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + 0$$

$$\Rightarrow f(x) = -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx$$

3. Find the fourier series expansion of $f(x) = x + x^2$ in the range $(-\pi, \pi)$.

Ans:- Here $c + 2l = \pi \Rightarrow -\pi + 2l = \pi$

$2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cdot\cos\left(\frac{n\pi x}{\pi}\right) + b_n\sin\left(\frac{n\pi x}{\pi}\right)\right]$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{-\pi^3}{3}\right)\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]$$

$$\cancel{= 2\pi^3/3} \quad \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{odd}}{x}\,\underset{\text{even}}{\cos(nx)}}_{\text{odd} = 0}\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{even}}{x^2}\cdot\underset{\text{even}}{\cos(nx)}}_{\text{even}}\,dx.$$

$$= \frac{1}{\pi}\times 0 + \frac{1}{\pi}\cdot 2\int_{0}^{\pi}x^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[x^2\cdot\frac{\sin(nx)}{n} - 2x\,\frac{-\cos(nx)}{n^2} - 2\cdot\frac{-\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} + \frac{2\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{2}{\pi}\left[\left(0+\frac{2.2\pi\cos 2\pi n}{n^2}+0\right)-(0+0+0)\right]$$

$$= \frac{2}{\pi}\times\frac{4\pi\cos 2n\pi}{n^2} = \frac{8.(-1)^{2n}}{n^2} = \underline{\underline{8/n^2}}. \quad n \neq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{odd}}{x}\,\underset{\text{odd}}{\sin(nx)}}_{\text{even}}\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{even}}{x^2}\cdot\underset{\text{odd}}{\sin(nx)}}_{\text{odd} = 0.}\,dx.$$

$$= \frac{1}{\pi}\times 2\int_{0}^{\pi}x\sin(nx)\,dx + 0$$

$$= \frac{2}{\pi}\left[x\cdot\frac{-\cos(nx)}{n} - 1\cdot\frac{-\sin(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos(nx)}{n}+\frac{\sin(nx)}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(-\pi\,\frac{\cos n\pi}{n}+0\right)-(0)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi\,(-1)^n}{n}\right] = \frac{-2.(-1)^n}{n} = \frac{2.(-1)^{n+1}}{n.}$$

$$\therefore f(x) = \frac{2\pi^3/3}{2} + \sum_{n=1}^{\infty}\left[\frac{8}{n^2}\cdot\cos(nx) + \frac{2\cdot(-1)^{n+1}}{n}\cdot\sin(nx)\right]$$

$$\Rightarrow f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty}\left[\frac{8\cos(nx)}{n^2} + \frac{2(-1)^{n+1}\sin(nx)}{n}\right]$$

4. Find the fourier Series of $f(x) = x^2 - 2$ in the interval $(-2, 2)$.

Ans: Here $c + 2l = 2 \Rightarrow -2 + 2l = 2 \Rightarrow 2l = 4 \Rightarrow l = 2$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\left(\frac{n\pi x}{2}\right)\right]$$

where $a_0 = \frac{1}{2}\int_{-2}^{2}(x^2-2)\,dx$

$$= \frac{1}{2}\left[\frac{x^3}{3} - 2x\right]_{-2}^{2} = \frac{1}{2}\left[\left(\frac{8}{3} - 4\right) - \left(\frac{-8}{3} + 4\right)\right]$$

$$= \frac{1}{2}\left[\frac{8}{3} - 4 + \frac{8}{3} - 4\right] = \frac{1}{2}\left[\frac{16}{3} - 8\right]$$

$$\frac{16-24}{3}$$

$$= \frac{1}{2} \times -8/3 = \underline{\underline{-4/3}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2}(x^2-2)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2}\underbrace{\underset{\text{even}}{x^2}\cdot\underset{\text{even}}{\cos\left(\frac{n\pi x}{2}\right)}}_{\text{even}}dx - \frac{1}{2}\int_{-2}^{2} 2\,\underset{\text{even}}{\cos\left(\frac{n\pi x}{2}\right)}dx$$

$$= \frac{1}{2}\times 2\int_{0}^{2} x^2\cos\left(\frac{n\pi x}{2}\right)dx - \frac{2}{2}\times 2\int_{0}^{2}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{2} x^2\cos\left(\frac{n\pi x}{2}\right)dx - 2\int_{0}^{2}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[x^2\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - 2x\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + 2\cdot\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$-2\cdot\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$= \left[x^2 \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} + 2x \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} - 2 \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} \right]_0^2$$

$$- 2\left[\frac{\sin\left(\frac{n\pi 2}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{\sin 0}{\left(\frac{n\pi}{2}\right)} \right]$$

$$= \left[2^2 x \cdot \frac{\sin(n\pi)}{\left(\frac{n\pi}{2}\right)} + 2\times 2 \frac{\cos(n\pi)}{\left(\frac{n\pi}{2}\right)^2} - 2 \frac{\sin(n\pi)}{\left(\frac{n\pi}{2}\right)^3} \right] -$$

$$\left[0 + 0 - 0 \right] - 2\left[0 - 0 \right]$$

$$= 0 + 4 \frac{\cos(n\pi)}{\left(\frac{n\pi}{2}\right)^2} - 2\times 0 - 0$$

$$= \frac{4(-1)^n}{\left(\frac{n\pi}{2}\right)} = 4(-1)^n \times \frac{2}{n\pi} = \underline{\underline{\frac{8(-1)^n}{n\pi}}} \text{ for } n \neq 0.$$

$$b_n = \frac{1}{2} \int_{-2}^{2} (x^2 - 2) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2} \int_{-2}^{2} \underset{\text{even}}{x^2} \cdot \underset{\text{odd}}{\sin\left(\frac{n\pi x}{2}\right)} dx - \frac{1}{2} \int_{-2}^{2} 2 \underset{\text{odd}}{\sin\left(\frac{n\pi x}{2}\right)} dx.$$

odd

$$= \frac{1}{2} \times 0 - \frac{1}{2} \times 0 = 0$$

$$\therefore f(x) = \frac{-4/3}{2} + \sum_{n=1}^{\infty} \left[\frac{8\cdot(-1)^n}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + 0 \right]$$

$$= \frac{-2}{3} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n\pi} \cdot \cos\left(\frac{n\pi x}{2}\right)$$

5. Find the Fourier Series representation of the Periodic function f(x) is given by:-

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \pi \\ 2\pi - x & \text{for } \pi < x \leq 2\pi. \end{cases}$$

Ans:- Here the interval is $[0, 2\pi]$

$$\Rightarrow 2l = 2\pi \Rightarrow l = \pi$$

Now, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)\right]$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + \left(\left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 \mp \frac{4\pi^2}{2} \mp 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{4\pi^2}{2}} + \cancel{2\pi^2}\right] = \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right] = \frac{1}{\pi} \times \pi^2 = \underline{\underline{\pi}}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx.$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cdot\cos(nx)\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos(n\pi)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cos(n\pi)\,dx + \int_{\pi}^{2\pi} 2\pi\cos(nx)\,dx \mp \int_{\pi}^{2\pi} x\cos(nx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\cancel{\left[x\frac{\cos n\pi}{n} - 1\cdot\frac{-\sin(n\pi)}{n^2}\right]_0^{\pi}} - 2\pi\cdot\cancel{\left[\frac{\sin(n\pi)}{n}\right]_{\pi}^{2\pi}}\right.$$

$$\cancel{+\left[x\right.}$$

$$= \frac{1}{\pi}\left\{\left[x\cdot\frac{\sin(nx)}{n} - 1\,\frac{-\cos(nx)}{n^2}\right]_0^{\pi} + 2\pi\left[\frac{\sin(nx)}{n}\right]_{\pi}^{2\pi}\right.$$

$$\left.\mp\left[x\,\frac{\sin(nx)}{n} - 1\cdot\frac{-\cos(nx)}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{ \left[0 + \frac{\cos n\pi}{n^2}\right] - \left[0 + \frac{\cos 0}{n^2}\right] + 2\pi\left[0 - 0\right] - \left\{\left[0 + \frac{\cos 2n\pi}{n^2}\right] - \left[0 + \frac{\cos 2\pi}{n^2}\right]\right\} \right\}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{(-1)^{2n}}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$bn = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx.$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(nx)dx + \int_{\pi}^{2\pi}(2\pi - x)\sin(nx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\frac{-\cos(nx)}{n} - 1.\frac{-\sin(nx)}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{-\cos(nx)}{n} - (-1)\frac{-\sin(nx)}{n^2}\right]_{\pi}^{2\pi}\right.$$

$$= \frac{1}{\pi}\left\{\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi} + \left[-(2\pi - x)\frac{\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi}^{2\pi}\right.$$

$$= \frac{1}{\pi}\left\{\left[-\pi\frac{\cos n\pi}{n} + 0\right] - 0 + \left[\left(0 - \left(-(2\pi - \pi)\frac{\cos n\pi}{n}\right) - 0\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi(-1)^n}{n} + \pi\frac{(-1)^n}{n}\right\}$$

$$= \frac{1}{\pi}\times\pi\left\{-\frac{(-1)^n}{n} + \frac{(-1)^n}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos(nx) + 0\right]$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos(nx)$$

6. Fourier Series expansion of an Even function in $(-l, l)$ or Fourier cosine Series in $(0, l)$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)} \qquad \because b_n = 0.$$

in $(0, l)$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$.

$$\text{where } a_0 = \frac{1}{l}\int_{-l}^{l} \underset{\text{even}}{f(x)}\,dx = \frac{2}{l}\int_0^l f(x)\,dx.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underset{\text{even}}{f(x)}\ \underset{\text{even}}{\cos\left(\frac{n\pi x}{l}\right)}}_{\text{even}}\,dx = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underset{\text{eddn}}{f(x)}\ \underset{\text{odd}}{\sin\left(\frac{n\pi x}{l}\right)}}_{\text{odd}}\,dx = 0.$$

Fourier Series expansion of an odd function in $(-l, l)$ or Fourier Sine Series in $(0, l)$.

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)} \qquad a_0 = 0, \quad \because a_n = 0.$$

$$\text{where } a_0 = \frac{1}{l}\int_{-l}^{l} \underset{\text{odd}}{f(x)}\,dx = 0.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underset{\text{odd}}{f(x)}\ \underset{\text{even}}{\cos\left(\frac{n\pi x}{l}\right)}}_{\text{odd}}\,dx = 0.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underset{\text{odd}}{f(x)}\ \underset{\text{odd}}{\sin\left(\frac{n\pi x}{l}\right)}}_{\text{even}}\,dx = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Que 1:- Develop the fourier Series of $f(x) = x^2$ in the interval $-2 \le x \le 2$.

Ans:- Here $f(x) = x^2$ is an even function

$\Rightarrow b_n = 0$

Period, $c + 2l = 2 \Rightarrow -2 + 2l = 2 \Rightarrow l = 2$

$$a_0 = \frac{2}{2}\int_0^2 x^2\,dx = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \underline{\underline{8/3}}.$$

$$a_n = \frac{2}{2}\int_0^2 x^2 \cdot \cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[x^2 \cdot \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - 2x\,\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + 2\,\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^3}\right]_0^2$$

$$= \left[2^2 \cdot \frac{\sin(n\pi)}{\left(\frac{n\pi}{2}\right)} + 2\times 2 \cdot \frac{\cos(n\pi)}{\left(\frac{n\pi}{2}\right)^2} - 2 \cdot \frac{\sin(n\pi)}{\left(\frac{n\pi}{2}\right)^3}\right] - \left[0 - 0 + 0\right]$$

$$= 0 + 4\,\frac{\cos(n\pi)}{\left(\frac{n\pi}{2}\right)^2} - 0 = 4\,(-1)^n \times \frac{4}{n^2\pi^2} = \underline{\underline{\frac{16(-1)^n}{n^2\pi^2}}}.$$

$$\therefore f(x) = \frac{8/3}{2} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right).$$

$$\Rightarrow f(x) = 4/3 + \sum_{n=1}^{\infty}\left[\frac{16(-1)^n}{n^2\pi^2}(\cos\left(\frac{n\pi x}{2}\right)\right]$$

2. Prove that in the interval $(-\pi, \pi)$ or $(0, 2\pi)$ fourier series of $f(x) = x\cos x$ with period 2π is

$$f(x) = \frac{-1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2-1}\sin(nx)$$

Ans:- $f(x) = \underbrace{x}_{\text{odd}}\,\underbrace{\cos x}_{\text{even}} \Rightarrow$ odd function

$\therefore a_0 = 0,\ a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) \Rightarrow f(x) = \underline{\underline{\sum_{n=1}^{\infty} b_n \cdot \sin(nx)}}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \cdot \sin\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos x \cdot \sin(nx)\,dx.$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\left[\sin(x+nx) - \sin(x-nx)\right]dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\sin(1+n)x\,dx - \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\sin(1-n)x\,dx.$$

$$= \frac{1}{\pi}\left[x \cdot \frac{-\cos(1+n)x}{1+n} - 1 \cdot \frac{-\sin(1+n)x}{(1+n)^2}\right]_0^{\pi} - \frac{\overset{1}{2}}{\pi}\left[x \frac{-\cos(1-n)x}{1-n} - 1 \cdot \frac{-\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$$

$$= \frac{\overset{1}{2}}{\pi}\left[-x\frac{\cos(1+n)x}{1+n} + \frac{\sin(1+n)x}{(1+n)^2}\right]_0^{\pi}$$

$$- \frac{\overset{1}{2}}{\pi}\left[-\frac{x\cos(1-n)x}{1-n} + \frac{\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$$

$$= \frac{\overset{1}{2}}{\pi}\left[\left(-\pi\frac{\cos(1+n)\pi}{(1+n)}\right) + 0 - (0+0)\right] - \frac{2}{\pi}\left[-\pi\frac{\cos(1-n)\pi}{(1-n)} + 0 - 0\right]$$

$$= \frac{\overset{1}{2}}{\pi} \times \frac{-\pi(-1)^{1+n}}{1+n} - \frac{\overset{1}{2}}{\pi} \times \frac{-\pi \cos(-1)^{1-n}}{(1-n)}$$

$$= -\frac{2(-1)^{1+n}}{1+n} + 2\frac{(-1)^{1-n}}{1-n}$$

$$= 2\left[\frac{-2 \times -1 \times (-1)^n}{1+n} + \frac{2 \times (-1) \times (-1)^{-n}}{1-n}\right]$$

$$= \frac{2(-1)^n}{1+n} - \frac{2(-1)^{-n}}{1-n} = ~~\frac{2(-1)^n - 2n(-1)^n - 2(-1)^{-n} - 2n(-1)^n}{(1+n)(1-n)}~~$$

$$= \frac{2(-1)^n}{1+n} - \frac{2 \times \left[(-1)^{-1}\right]^n}{1-n} = \frac{2(-1)^n}{1+n} - \frac{2 \times \left(\frac{1}{-1}\right)^n}{1-n}$$

$$= \frac{2(-1)^n}{1+n} - \frac{2 \cdot (-1)^n}{1-n}$$

$$= \frac{2(-1)^n - 2n(-1)^n - 2(-1)^n - 2n(-1)^n}{(1+n)(1-n)}$$

$$= \frac{-2n(-1)^n}{1-n^2} = \frac{2n(-1)^n}{n^2-1} \quad n \neq 1$$

When, $n=1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \cdot \sin x\, dx.$$ $2\sin x \cos x = \sin 2x$

$$= \frac{2}{\pi}\int_0^{\pi} x\cdot\frac{\sin 2x}{2}dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[x\,\frac{-\cos 2x}{2} - 1\cdot\frac{-\sin 2x}{4}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos 2\pi}{2} + 0\right] - \left[0+0\right]$$

$$= \frac{1}{\pi}\times\frac{-\pi(-1)^2}{2} = -1/2$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin(nx)$$

$$= -1/2 \sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1}\sin(nx)$$

3. Find the half range cosine series expansion for $f(x) = x^2$ in $0 \le x \le \pi$

Ans: Here $f(x) = x^2$ in $0 \le x \le \pi$, Period $= \cancel{2l} = 2\pi$
$\Rightarrow l = \pi$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx$

$$= \frac{2}{\pi}\times\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\times\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)dx$$

$$= \left[x^2\,\frac{\sin(nx)}{n} - 2x\,\frac{-\cos(nx)}{n^2} + 2\,\frac{-\sin(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - 2\frac{\sin(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos n\pi}{n^2} - 0\right] - \left[0 + 0 - 0\right)$$

$$= \frac{2}{\pi} \times \frac{2\pi\,(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}.$$

$$\therefore f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cdot\cos(nx)$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

Que 4:- Develop the half range cosine Series of $f(x) = \pi x - x^2$ in the interval $0 < x < \pi$.

Ans:- Period $l = \pi$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(nx)$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^{\pi}(\pi x - x^2)\,dx$.

$$= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi\cdot\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\times\frac{\pi^3}{6} = \pi^2/3$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos(nx)\,dx$$

$$\frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin(nx)}{n} - (\pi - 2x)\frac{-\cos(nx)}{n^2} + (0-2)\frac{-\sin(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin(nx)}{n} + (\pi - 2x)\frac{\cos(nx)}{n^2} + 2\frac{\sin(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + (\pi - 2\pi)\frac{\cos n\pi}{n^2} + 0\right] - \left[0 + (\pi - 0)\frac{\cos 0}{n^2} + 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cdot(-1)^n}{n^2} - \pi\cdot\frac{1}{n^2} \right] = \frac{2}{\pi}\left[\pi \times \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] \right.$$

$$= \frac{2\left(\left(-1\right)^n - 1\right)}{n^2}$$

$$\therefore f(x) = \frac{\frac{\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2} \cdot \cos(nx)$$

$$\Longrightarrow f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2} \cos(nx)$$

6. Find the fourier cosine series of $f(x) = x \sin x$ in $0 < x < \pi$.

Ans:- $l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{\pi}\right)$$

$$\Longrightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$

$$= \frac{2}{\pi}\left[x \, {}^{-}\cos x - 1 \cdot {}^{-}\sin x \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x \right]_0^{\pi} = \frac{2}{\pi}\left[-\pi\cos\pi + 0 \right] - \left[0 + 0 \right]$$

$$= \frac{2}{\pi} \times -\pi \times (-1) = 2.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot \cos(nx)\,dx.$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B) \right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x \times \frac{1}{2}\left[\sin(x+nx) + \sin(x-nx) \right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin(1+n)x \, dx + \frac{1}{\pi}\int_0^{\pi} x \sin(1-n)x \, dx$$

$$= \frac{1}{\pi}\left[x \, \frac{{}^{-}\cos(1+n)x}{1+n} - 1 \cdot \frac{{}^{-}\sin(1+n)x}{1+n} \right]_0^{\pi} + \frac{1}{\pi}\left[x \, \frac{{}^{-}\cos(1-n)x}{1-n} - 1 \cdot \frac{{}^{-}\sin(1-n)x}{1-n} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos(1+n)x}{1+n} + \frac{\sin(1+n)x}{1+n} \right]_0^{\pi} + \frac{1}{\pi}\left[\frac{-x\cos(1-n)x}{1-n} + \frac{\sin(1-n)x}{1-n} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n} + 0 - (0+0)\right] + \frac{1}{\pi}\left[\frac{-\pi\cos(1-n)\pi}{1-n} + 0 - (0+0)\right]$$

$$= \frac{1}{\pi} \times \frac{-\pi\,(-1)^{1+n}}{1+n} + \frac{1}{\pi} \times \frac{-\pi\,(-1)^{1-n}}{1-n}$$

$$= \frac{-1 \times (-1)^1 \times (-1)^n}{1+n} + \frac{(-1)\,(-1)^1\,(-1)^{-n}}{1-n}$$

$$= \frac{-1 \times -1 \times (-1)^n}{1+n} + \frac{(-1)(-1)(-1)^n}{1-n}$$

$\left(\frac{1}{-1}\right)^n = (-1)^n$

$$= \frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} = (-1)^n\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= (-1)^n\left[\frac{1-n+1+n}{(1+n)(1-n)}\right] = \frac{(-1)^n \times 2}{1-n^2} = \cancel{\frac{2(-1)^n}{n^2-1}}$$

$$= \frac{2(-1)^n}{1-n^2} \quad n \neq 1$$

~~$\therefore f(x) = \frac{2}{2} + \frac{1}{2}$~~

when $n=1$, $a_1 = \frac{2}{\pi}\int_0^\pi x \sin x \cos x \, dx$

$$= \frac{2}{\pi} \times \int_0^\pi x \cdot \frac{\sin 2x}{2}\, dx.$$

$$= \frac{1}{\pi}\int_0^\pi x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x \cdot \frac{-\cos 2x}{2} - 1 \cdot \frac{-\sin 2x}{2}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos 2\pi}{2} + 0 - (0+0)\right]$$

$$= \frac{1}{\pi} \times \frac{-\pi \times 1}{2} = -1/2$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) dx$$

$$= \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos(nx) dx$$

$$\Rightarrow f(x) = 1 + -1/2\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2}\cos(nx)$$

6. Find the fourier Sine Series of $f(x) = e^x$ in $0<x<1$.

Ans: $l = 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right) \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(n\pi x)$$

where $b_n = \frac{2}{1}\int_0^1 e^x \cdot \sin(n\pi x)\,dx$.

$$= \left[2 \cdot \frac{e^x}{1+(n\pi)^2} \cdot [1 \cdot \sin(n\pi x) - n\pi \cos(n\pi x)]\right]_0^1$$

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$= 2 \cdot \left\{ \frac{e^1}{1+n^2\pi^2}\left[\sin(n\pi) - n\pi\cos(n\pi)\right] - \left\{\frac{e^0}{1+n^2\pi^2}\left[0 - n\pi \cos 0\right]\right\} \right.$$

$$= 2 \cdot \left\{ \frac{e}{1+n^2\pi^2}\left[0 - n\pi(-1)^n\right] + \frac{n\pi}{1+n^2\pi^2} \right\}$$

$$= \frac{2e}{1+n^2\pi^2} \times \left[-n\pi(-1)^n\right] + \frac{2n\pi}{1+n^2\pi^2}$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[-e\,(-1)^n + 1\right] = \frac{2n\pi}{1+n^2\pi^2}\left[1 - e(-1)^n\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right) \cdot \sin(n\pi x)$$

7. Find the fourier sine series of $f(x) = \begin{cases} x, & 0<x<2 \\ 4-x, & 2<x<4 \end{cases}$

Ans: Here, $(0, l) = (0, 4) \Rightarrow l = 4$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

where $b_n = \frac{2}{4}\int_0^4 f(x) \sin\left(\frac{n\pi x}{4}\right) dx$

$$= \frac{1}{2}\left\{ \int_0^2 x \cdot \sin\left(\frac{n\pi x}{4}\right) dx + \int_2^4 (4-x) \sin\left(\frac{n\pi x}{4}\right) dx \right\}$$

$$= \frac{1}{2}\left\{ \left[x \frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} - 1 \cdot \frac{-\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2} \right]_0^2 + \right.$$

$$\left. \left[(4-x) \frac{-\cos\left(\frac{n\pi x}{4}\right)}{(n\pi/4)} - (0-1) \frac{-\sin\left(\frac{n\pi x}{4}\right)}{(n\pi/4)^2} \right]_2^4 \right\}$$

$$\frac{1}{2}\left\{\left[-x\frac{\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}+\frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]_0^2+\left[-(4-x)\frac{\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}-\frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]_2^4\right.$$

$$=\frac{1}{2}\left\{\left[\frac{-2\cos\left(\frac{n\pi\times2}{4}\right)}{\left(\frac{n\pi}{4}\right)}+\frac{\sin\left(\frac{n\pi\times2}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]-0\right.$$

$$+\left[-(4-4)\frac{\cos\left(\frac{n\pi\times4}{4}\right)}{\left(\frac{n\pi}{4}\right)}-\frac{\sin\left(\frac{n\pi\times4}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]-\left[-(4-2)\frac{\cos\left(\frac{n\pi 2}{4}\right)}{\left(\frac{n\pi}{4}\right)}-\frac{\sin\left(\frac{n\pi\times2}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]$$

$$=\frac{1}{2}\left(\frac{-2\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)}+\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)^2}+0-0+2\cdot\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)}+\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)^2}\right.$$

$$=\frac{1}{2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)^2}+\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]$$

$$=\frac{1}{2}\times\frac{2\sin\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2}{16}}=\frac{16\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\frac{16\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\cdot\sin\left(\frac{n\pi x}{4}\right)$$

8. Find the fourier sine series of $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2.\end{cases}$

Ans:- $l=2$. $(0,2)$.

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{2}\right).$$

where $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

$$= \int_0^1 x \sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[x \frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 1\cdot\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^1 + \left[(2-x)\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - (0-1)\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_1^2$$

$$= \left[1\cdot\frac{-\cos\left(\frac{\pi n}{2}\right)}{\left(\frac{n\pi}{2}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right] - 0 + \Big[(0-0) - \Big[(2-1)\frac{-\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\Big]$$

$$= \frac{-\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}$$

$$= \frac{2\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} = 2\sin\left(\frac{n\pi}{2}\right) \times \frac{4}{n^2\pi^2} = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}.$$

$$\therefore f(x) = \underline{\underline{\sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right).\sin\left(\frac{n\pi x}{2}\right)}}$$