Module 3:- Syllabus Discription (9 hrs)

Laplace transform, Inverse Laplace Transform, Linearity property, First Shifting theorem. Transform of derivatives. Solution of initial Valued Problem by Laplace transform (Second order linear ODE with constant coefficients with initial condition at $t=0$ only) Unit step function, Second Shifting theorem. Dirac delta function and its transform (Initial value problem involving Unit step function and Dirac delta function are excluded) Convolution theorem (without proof) and its application to finding inverse Laplace transform of product of functions.

Applications:- Laplace transform is very useful to obtaining solution of Linear Differential equations, both ordinary and partial, Solutions of System of Simultaneous Differential Equations, Solution of Integral Equations, Solutions of Linear differential equations.

* Particular solution is obtained without first determining the general Solution.
* Non-Homogeneous Equations are solved without obtaining the complementary integral.

Definition of Laplace Transformation

Let $f(t)$ be a given function defined for all $t \geq 0$, then the Laplace transformation of $f(t)$ is defined as,

$$\boxed{L[f(t)] = F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt}$$

Here L is Called the Laplace Transform Operator. The function $f(t)$ is known as the determining function, depends on t.

The new function which is to be determined [ie, $F(s)$] is called generating function, depends on s.

Properties of Laplace Transformation

1) $L[k] = \frac{k}{s}$ where k is a constant.

2) $L[1] = \frac{1}{s}$

3) $L[e^{at}] = \frac{1}{s-a}$ and $L[e^{-at}] = \frac{1}{s+a}$

4) $L[\cos at] = \frac{s}{s^2+a^2}$

5) $L[\sin at] = \frac{a}{s^2+a^2}$

6) $L[\cosh at] = \frac{s}{s^2-a^2}$

7) $L[\sinh at] = \frac{a}{s^2-a^2}$

8) $L[t^n] = \frac{n!}{s^{n+1}}$

Linear Properties of Laplace Transformation

If a and b are constants and $g_1(t)$ and $g_2(t)$ are given functions, then

$$L[a g_1(t) + b g_2(t)] = a \, L[g_1(t)] + a_2 L[g_2(t)]$$

First Shifting Theorem

If $L[f(t)] = F(s)$, then $\boxed{L[e^{at} f(t)] = F(s-a) = [F(s)]_{s \to s-a}}$

Similarly, if $L[f(t)] = F(s)$, then

$$\boxed{L[e^{-at} f(t)] = F(s+a) = [F(s)]_{s \to s+a}}$$

Que1:- find $L[e^{-at} \cdot \sin bt] \longrightarrow (1)$

Ans:- By first shifting theorem,

$$L[e^{-at} f(t)] = [F(s)]_{s \to s+a} \longrightarrow (2)$$

comparing,

$f(t) = \sin bt$.

$$F(s) = L[f(t)] = L[\sin bt]$$

$$= \frac{b}{s^2+b^2}$$

Now,

$$L[e^{-at} \sin bt] = \left[\frac{b}{s^2+b^2}\right]_{s \to s+a}$$

$$\Rightarrow L[e^{-at} \sin bt] = \frac{b}{(s+a)^2+b^2}$$

2) Find $L[e^{2t} \cdot \cos 5t]$

Ans:- Here $f(t) = \cos 5t$.

$$F(s) = L[f(t)] = L[\cos 5t]$$

$$= \frac{s}{s^2+5^2} = \frac{s}{s^2+25}$$

$$\therefore L[e^{2t} \cos 5t]$$

$$= [F(s)]_{s \to s-2}$$

$$= \left[\frac{s}{s^2+25}\right]_{s \to s-2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

3) Find $L[e^{-3t} \cdot \sin 3t]$

Ans:- Here $f(t) = \sin 3t$.

$$F(s) = L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{s^2+3^2}$$

$$\therefore L[e^{-3t} \cdot \sin 3t] = (F(s))_{s \to s+3}$$

$$= \left[\frac{3}{s^2+9}\right]_{s \to s+3}$$

$$= \frac{3}{(s+3)^2+9}$$

Que 4:- Find $L[t^2 \cdot e^{-2t}]$

Ans:- Here $f(t) = t^2$.

$$F(s) = L[f(t)] = L[t^2]$$

$$= \frac{2!}{s^{2+1}} = \frac{2}{s^3}.$$

$$\therefore L[t^2 \cdot e^{-2t}] = [F(s)]_{s \to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s \to s+2}$$

$$= \frac{2}{(s+2)^3}.$$

Que 5:- find $L[e^{-3t} \cos 2t]$.

Ans:- Here $f(t) = \cos 2t$.

$$F(s) = L[f(t)] = L[\cos 2t]$$

$$= \frac{s}{s^2+2^2} = \frac{s}{s^2+4}.$$

$$\therefore L[e^{-3t} \cos 2t] = [F(s)]_{s \to s+3}$$

$$= \left(\frac{s}{s^2+4}\right)_{s \to s+3}$$

$$= \frac{(s+3)}{(s+3)^2+4}$$

Que 6:- Find $L[e^{-2t}(3\sinh 2t - 5\cosh 2t)]$

Ans:- Here $f(t) = 3\sinh 2t - 5\cosh 2t$

$F(s) = L[f(t)] = L[3\sinh 2t - 5\cosh 2t]$

$= 3 \cdot L[\sinh 2t] - 5\, L[\cosh 2t]$

$= 3 \cdot \frac{2}{s^2-2^2} - 5 \cdot \frac{s}{s^2-2^2}$

$= \frac{6}{s^2-4} - \frac{5s}{s^2-4}$

$\therefore L[e^{-2t}(3\sinh 2t - 5\cosh 2t)] = [F(s)]_{s \to s+2}$

$= \left[\frac{6}{s^2-4} - \frac{5s}{s^2-4}\right]_{s \to s+2}$

$= \left(\frac{6-5s}{s^2-4}\right]_{s \to s+2} = \frac{6-5(s+2)}{(s+2)^2-4}$

$= \frac{6-5s-10}{(s+2)^2-4} = \frac{-4-5s}{(s+2)^2-4}$

Que 7:- Find $L[e^{2t} + 2e^{-4t}]$

Ans:- $= L[e^{2t}] + 2 \cdot L[e^{-4t}]$

$= \frac{1}{s-2} + 2 \cdot \frac{1}{s+4}$

$= \frac{1}{s-2} + \frac{2}{s+4}$

Que 8:- Find $L[e^{-3t}\sin^2 t]$

Ans:- Here $f(t) = \sin^2 t$

$\Rightarrow f(t) = \frac{1-\cos 2t}{2}$

$\Rightarrow f(t) = \frac{1}{2} - \frac{\cos 2t}{2}$

$F(s) = L[f(t)] = L\left[\frac{1}{2} - \frac{\cos 2t}{2}\right]$

$= \frac{1}{2} L[1] - \frac{1}{2} L[\cos 2t]$

$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2+2^2}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$\therefore L[e^{-3t}\sin^2 t] = [F(s)]_{s \to s+3}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]_{s \to s+3}$

$= \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right]$

Que 9:- Find $L[e^{3t} \cdot \sin^2 2t]$

Ans:- Here $f(t) = \sin^2 2t$.

$$\Rightarrow f(t) = \frac{1-\cos 4t}{2}$$

$$L[f(t)] = \frac{1}{2} L[1-\cos 4t]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$\therefore L[e^{3t} \cdot \sin^2 2t] = \left(F(s)\right)_{s \to s-3}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]_{s \to s-3}$$

$$= \frac{1}{2}\left[\frac{1}{s-3} - \frac{s-3}{(s-3)^2+16}\right]$$

Que 10:- Solve $L[\cos^3 2t]$

Ans:-

$$L[\cos^3 2t] = L\left[\frac{\cos 6t + 3\cos 2t}{4}\right]$$

$$= \frac{1}{4}\left\{L[\cos 6t] + L[3\cos 2t]\right\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+6^2} + 3 \cdot \frac{s}{s^2+2^2}\right\}$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$$

Que 11:- Solve $L[\sin^3 3t]$

$$= L\left[\frac{3\sin 3t - \sin 9t}{4}\right]$$

$$= \frac{1}{4}\left\{L[3\sin 3t] - L[\sin 9t]\right\}$$

$$= \frac{1}{4}\left[3 \cdot \frac{3}{s^2+3^2} - \frac{9}{s^2+9^2}\right] = \frac{1}{4}\left[\frac{9}{s^2+9} - \frac{9}{s^2+81}\right]$$

Results:-

1) $\cos^2 x + \sin^2 x = 1$
2) $\cos 2x = 1 - 2\sin^2 x$
   $\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$
3) $\cos 2x = 2\cos^2 x - 1$
   $\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$
4) $\sin 2x = 2\sin x \cos x$
5) $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$
6) $\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$
7) $\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$
8) $\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$
9) $\cos 3x = 4\cos^3 x - 3\cos x$
   $\cos^3 x = \frac{\cos 3x + 3\cos x}{4}$
10) $\sin 3x = 3\sin x - 4\sin^3 x$
    $\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

Que 12:- Find $L\left[e^{2t}\cdot \sin 3t\cdot \cos 2t\right]$

Ans:- Here $f(t) = \sin 3t \cdot \cos 2t$ $\qquad \sin A \cos B = \dfrac{\sin(A+B) + \sin(A-B)}{2}$

$$\Rightarrow f(t) = \frac{\sin(5t) + \sin t}{2}$$

$$\Rightarrow L[f(t)] = F(s) = \frac{1}{2}\left\{ L[\sin 5t] + L[\sin t]\right\}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} + \frac{1}{s^2+1^2}\right] = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$\therefore L\left[e^{2t}\cdot \sin 3t \cdot \cos 2t\right] = \left[F(s)\right]_{s\to s-2}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]_{s\to s-2} = \frac{1}{2}\left[\frac{5}{(s-2)^2+25} + \frac{1}{(s-2)^2+1}\right]$$

Que 13:- Find $L\left[e^{-3t}\cdot \sin 5t \cdot \sin 3t\right]$

Ans:- Here $f(t) = \sin 5t \cdot \sin 3t$

$$F(s) = L[f(t)] = L[\sin 5t \cdot \sin 3t] \qquad \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$= L\left[\frac{\cos 3t - \cos 7t}{2}\right] =$$

$$= \frac{1}{2}\left\{L[\cos 3t] - L[\cos 7t]\right\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+3^2} - \frac{s}{s^2+7^2}\right]$$

$$\therefore L\left[e^{-3t}\cdot \sin 5t \cdot \sin 3t\right] = \left[F(s)\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+3^2} - \frac{s}{s^2+49}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+9} - \frac{s+3}{(s+3)^2+49}\right]$$

Laplace Transform of Derivatives

If $f(t)$ is continious and $f'(t)$ also continious then the Laplace transform of $f'(t)$ is given by

$$L[f'(t)] = sL[f(t)] - f(0)$$

Result :- ① Laplace transform of $t\,f(t)$ or
If $L[f(t)] = F(s)$, then $\boxed{L[t\,f(t)] = -\frac{d}{ds}F(s).}$

Result ② :- If $L[f(t)] = F(s)$, then

$$\boxed{L[t^n \cdot f(t)] = (-1)^n \cdot \frac{d^n}{ds^n} F(s)}$$

ie, $L[t^2 \cdot f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$,

$L[t^3 f(t)] = (-1)^3 \cdot \frac{d^3}{ds^3} F(s)$ and so on.

Que 1 :- Find $L[t \sin at]$?

Ans :- We have $L[t\,f(t)] = -\frac{d}{ds}F(s)$, $F(s) = L[f(t)]$

$$\Rightarrow L[t \sin at] = \frac{-d}{ds} L[\sin at]$$

$$= \frac{-d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\left[\frac{0 \times (s^2+a^2) - a(2s+0)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{-2as}{(s^2+a^2)^2}\right] = \frac{2as}{(s^2+a^2)^2}$$

Que 2:- Solve $L[t \sin 2t]$

Ans:- $L[t\, f(t)] = -\frac{d}{ds} F(s)$

$$L[t \sin 2t] = -\frac{d}{ds} L[\sin 2t]$$

$$= -\frac{d}{ds}\left[\frac{2}{s^2+2^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\left[\frac{0 \times (s^2+4) - 2(2s+0)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-4s}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$$

Que 3:- Find $L[t^2 \cdot e^{-2t}]$

Ans:- $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} L[f(t)]$

$$= \frac{d^2}{ds^2} L[f(t)]$$

$$L[t^2 e^{-2t}] = \frac{d^2}{ds^2} L[e^{-2t}]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s+2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{0 \times (s+2) - 1(1+0)}{(s+2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{0-1}{(s+2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+2)^2}\right] = -\frac{d}{ds}\left[\frac{1}{(s+2)^2}\right]$$

$$= -\left[\frac{0 \times (s+2)^2 - 2(s+2) \times 1 \times 1}{(s+2)^4}\right]$$

$$= -\left[\frac{-2(s+2)}{(s+2)^4}\right]$$

$$= \frac{2(s+2)}{(s+2)^4} = \frac{2}{(s+2)^3}$$

Que 4:- Find $L[4t\, e^{-2t}]$

Ans:- $L[t\, f(t)] = -\frac{d}{ds}[F(s)]$

$$F(s) = L[f(t)]$$

$$\therefore L[4t\, e^{-2t}] = 4 \cdot -\frac{d}{ds} L[f(t)]$$

$$= -4 \frac{d}{ds} L[e^{-2t}]$$

$$= -4 \frac{d}{ds}\left[\frac{1}{s+2}\right]$$

$$= -4\left[\frac{0 \times (s+2) - 1 \times (1+0)}{(s+2)^2}\right]$$

$$= -4\left[\frac{-1}{(s+2)^2}\right] = \frac{4}{(s+2)^2}$$

Que 5:- Solve $L[t \cos 3t]$

Ans:- $L[t\, f(t)] = -\frac{d}{ds} F(s)$

$$L[t \cos 3t] = -\frac{d}{ds} L[\cos 3t]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+9}\right]$$

$$= -\left[\frac{1 \times (s^2+9) - s(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right] = -\left[\frac{9-s^2}{(s^2+9)^2}\right]$$

$$= \frac{s^2-9}{(s^2+9)^2}$$

Que 6: Find $L[t\, e^{-t} \sin t]$

Ans: $L[t\, f(t)] = \frac{-d}{ds} L[f(t)]$.

Here ~~$L[f(t)]$~~ $f(t) = e^{-t} \sin t$.

$L[f(t)] = L[e^{-t} \sin t]$

$= L[\sin t]_{s \to s+1}$

$= \left[\frac{1}{s^2+1}\right]_{s \to s+1} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+1+1}$

$= \frac{1}{s^2+2s+2}$

Now, $L[t\, e^{-t} \sin t] = \frac{-d}{ds} L[e^{-t} \sin t]$

$= -\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right]$

$= -\left[\frac{0 \times (s^2+2s+1) - 1 \times (2s+2)}{(s^2+2s+2)^2}\right]$

$= \frac{2s+2}{(s^2+2s+2)^2}$

Que 7: Find $L[t\, e^{-t} \cosh t]$

Ans: $L[t\, f(t)] = \frac{-d}{ds} L[f(t)]$

Here $f(t) = e^{-t} \cosh t$.

$L[f(t)] = L[e^{-t} \cosh t]$

$= L[\cosh t]_{s \to s+1}$

$= \left[\frac{s}{s^2-1}\right]_{s \to s+1} = \frac{s+1}{(s+1)^2-1}$

$= \frac{s+1}{s^2+2s+1-1} = \frac{s+1}{s^2+2s}$.

$L[t\, e^{-t} \cosh t] = \frac{-d}{ds} L[f(t)]$

$= -\frac{d}{ds}\left[\frac{s+1}{s^2+2s}\right]$

$= -\left[\frac{(1+0)(s^2+2s) - (s+1)(2s+2)}{(s^2+2s)^2}\right]$

$= -\left[\frac{s^2+2s-(2s^2+2s+2s+2)}{(s^2+2s)^2}\right]$

$= -\left[\frac{s^2+2s-2s^2-4s-2}{(s^2+2s)^2}\right]$

$= -\left[\frac{-s^2-2s-2}{(s^2+2s)^2}\right]$

$= \frac{s^2+2s+2}{(s^2+2s)^2}$.

Que 8: Find $L\left[t^2 e^{2t} \cos 3t\right]$

Ans:- $L\left[t^2 f(t)\right] = (-1)^2 \dfrac{d^2}{ds^2} L\left[f(t)\right] = \dfrac{d^2}{ds^2} L\left[e^{2t} \cos 3t\right]$

$$L\left[e^{2t} \cdot \cos 3t\right] = L\left[\cos 3t\right]_{s \to s-2}$$

$$= \left[\frac{s}{s^2+9}\right]_{s \to s-2} = \frac{s-2}{(s-2)^2+9}$$

$$L\left[t^2 \cdot e^{2t} \cdot \cos 3t\right] = \frac{d^2}{ds^2}\left[\frac{s-2}{(s-2)^2+9}\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left[\frac{s-2}{(s-2)^2+9}\right]\right]$$

$$= \frac{d}{ds}\left[\frac{(1-0)\left((s-2)^2+9\right) - (s-2) \times \left(2(s-2) \times (1-0) - 0\right)}{\left[\left(s-2\right)^2+9\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s-2)^2+9 - (s-2) \times 2(s-2)}{\left((s-2)^2+9\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s-2)^2+9-2(s-2)^2}{\left((s-2)^2+9\right)^2}\right] = \frac{d}{ds}\left[\frac{9-(s-2)^2}{\left[(s-2)^2+9\right]^2}\right]$$

$$= \frac{d}{ds}\left[\frac{9-(s^2-4s+4)}{(s^2-4s+4+9)^2}\right] = \frac{d}{ds}\left[\frac{9-s^2+4s-4}{(s^2-4s+13)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2+4s+5}{(s^2-4s+13)^2}\right] =$$

$$= \frac{(-2s+4)(s^2-4s+13)^2 - (-s^2+4s+5) \times 2(s^2-4s+13)(2s-4)}{(s^2-4s+13)^4}$$

$$= \frac{(s^2-4s+13)\left((-2s+4) - 2(-2s^3+8s^2+10s-4s^2-16s-20\right.}{(s^2-4s+13)^4}$$

$$= \frac{(s^2-4s+13)(-2s+4+4s^3-16s^2-20s+8s^2+32s+40}{(s^2-4s+13)^4}$$

$$= \frac{(s^2-4s+13)(4s^3-8s^2+10s+44)}{(s^2-4s+13)^4}$$

$$=\frac{(s^2-4s+13)(2s-4)\left[-(s^2-4s+13)-2(-s^2+4s+5)\right]}{(s^2-4s+13)^4}$$

$$=\frac{(s^2-4s+13)(2s-4)\left[-s^2+4s-13+2s^2-8s-10\right]}{(s^2-4s+13)^4}$$

$$=\frac{(2s-4)(s^2-4s-23)}{(s^2-4s+13)^3}=\frac{2s^3-8s^2-46s-4s^2+16s+92}{(s^2-4s+13)^2}$$

$$=\frac{2s^3-12s^2-30s+92}{(s^2-4s+13)^3}$$

Inverse Laplace Transformation

	Laplace Transformation	Inverse Laplace Transformation.
1.	$L[1]=\frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right]=1$
2.	$L[e^{at}]=\frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
3.	$L[e^{-at}]=\frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}$
4.	$L[\sin at]=\frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$ or $L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}$
5.	$L[\cos at]=\frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$
6.	$L[\sinh at]=\frac{a}{s^2-a^2}$	$L^{-1}\left[\frac{a}{s^2-a^2}\right]=\sinh at$ or $L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$
7.	$L[\cosh at]=\frac{s}{s^2-a^2}$	$L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$
8.	$L[t^n]=\frac{n!}{s^{n+1}}$	$L^{-1}\left[\frac{n!}{s^{n+1}}\right]=t^n$ or $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$

Result:

$$L\left[e^{at}\sin bt\right] = L[\sin bt]_{s\to s-a}$$

$$= \left[\frac{b}{s^2+b^2}\right]_{s\to s-a} = \frac{b}{(s-a)^2+b^2}.$$

$$\Rightarrow L\left[e^{at}\sin bt\right] = \frac{b}{(s-a)^2+b^2}$$

① $L^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = e^{at}\sin bt$ or $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$

similarly,

② $L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right] = \frac{1}{b}e^{-at}\sin bt$.

③ $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$.

④ $L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] = e^{-at}\cos bt$.

Partial Fraction

	Factors of Denominator	Corresponding Partial fractions.
1.	Non-Repeated linear factors Eg:- $\frac{1}{s-a}$ [(s-a) occurs only one time]	$\frac{A}{s-a}$
	$\frac{1}{(s-a)(s-b)}$ [(s-a) and (s-b) all occurs only one time.	$\frac{A}{s-a} + \frac{B}{s-b}$.
	$\frac{1}{s(s-a)(s-b)}$	$\frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b}$.

2.	Repeated linear factors repeated r times :- Ex:- $(s-a)^r$	$\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\cdots+\frac{A_r}{(s-a)^r}$
	$\frac{1}{(s-a)^2}$	$\frac{A}{s-a}+\frac{B}{(s-a)^2}$
	$\frac{1}{(s-a)(s-b)^3}$	$\frac{A}{s-a}+\frac{B}{(s-b)}+\frac{C}{(s-b)^2}+\frac{D}{(s-b)^3}$
3.	Non Repeated Quadratic Expressions :- s^2+as+b	$\frac{As+B}{s^2+as+b}$, Here atleast one of A or $B \neq 0$.
4.	Repeated Quadratic Expression :- repeated r times :- Ex:- $(s^2+as+b)^r$	$\frac{A_1s+B_1}{s^2+as+b}+\frac{A_2s+B_2}{(s^2+as+b)^2}+\cdots+\frac{A_rs+B_r}{(s^2+as+b)^r}$
	Eg:- $\frac{1}{(s-a)(s^2+2s+3)}$	$\frac{A}{s-a}+\frac{Bx+C}{s^2+2s+3}$

<u>Que 1</u>:- Find $L^{-1}\left[\cancel{\frac{s}{(s^2+a^2)^2}}\right]$ $L^{-1}\left[\frac{s-2}{s^2+5s+6}\right]$ using partial fraction method.

<u>Ans</u>: Consider $s^2+5s+6=0 \Rightarrow s=-3,\ s=-2$.

$\therefore (s+3)(s+2)$ are the factors of s^2+5s+6.

$$\therefore \frac{s-2}{s^2+5s+6}=\frac{s-2}{(s+2)(s+3)}=\frac{A}{s+2}+\frac{B}{s+3}.$$

$$\Rightarrow \frac{s-2}{(s+2)(s+3)}=\frac{A(s+3)+B(s+2)}{(s+2)(s+3)}.$$

$$\Rightarrow s-2=A(s+3)+B(s+2)$$

Put $s=-3$

$$-3-2=A(-3+3)+B(-3+2)$$

$$\Rightarrow -5=0-B \Rightarrow B=5.$$

Put $s=-2$

$$-2-2=A(-2+3)+B(-2+2)$$

$$-4=A+0 \Rightarrow A=\underline{\underline{-4}}.$$

s^2+5s+6
$=(s+5/2)^2+6-\frac{25}{4}$
$=(s+5/2)^2-\frac{1}{4}$

$\frac{s-2}{(s+5/2)^2-1/4}$

$=\frac{s+5/2-5/2-2}{(s+5/2)^2-1/4}$

$L^{-1}\left[\frac{s+5/2}{(s+5/2)^2-\cdots}\right]-\cdots$

Now, $\dfrac{S-2}{S^2+5S+6} = \dfrac{S-2}{(S+2)(S+3)} = \dfrac{-4}{S+2} + \dfrac{5}{S+3}$

$$\therefore L^{-1}\left[\frac{S-2}{S^2+5S+6}\right] = L^{-1}\left[\frac{-4}{S+2} + \frac{5}{S+3}\right]$$

$$= -4\, L^{-1}\left[\frac{1}{S+2}\right] + 5\cdot L^{-1}\left[\frac{1}{S+3}\right]$$

$$= -4e^{-2t} + 5e^{-3t}$$

Que 2:- Find $L^{-1}\left[\dfrac{7S-1}{(S+1)(S+2)(S+3)}\right]$ using partial fraction.

Ans:- Consider $\dfrac{7S-1}{(S+1)(S+2)(S+3)} = \dfrac{A}{S+1} + \dfrac{B}{S+2} + \dfrac{C}{S+3}$.

$$\Rightarrow \frac{7S-1}{(S+1)(S+2)(S+3)} = \frac{A(S+2)(S+3) + B(S+1)(S+3) + C(S+1)(S+2)}{(S+1)(S+2)(S+3)}$$

$$\Rightarrow 7S-1 = A(S+2)(S+3) + B(S+1)(S+3) + C(S+1)(S+2).$$

Put $S=-1$.

$$-7-1 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$-8 = A(1)(2) + 0 + 0.$$

$$\Rightarrow -8 = 2A \Rightarrow A = -4.$$

Put $S=-2$

$$-14-1 = A(-2+2)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(-2+2)$$

$$-15 = 0 + B(-1)(+1) + 0.$$

$$-15 = -B \Rightarrow B = 15$$

Put $S=-3$

$$-21-1 = A(-3+2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3+2)$$

$$-22 = 0 + 0 + C(-2)(-1)$$

$$-22 = 2C \Rightarrow C = -11$$

$$\therefore L^{-1}\left[\frac{7S-1}{(S+1)(S+2)(S+3)}\right] = L^{-1}\left[\frac{-4}{S+1} + \frac{15}{S+2} + \frac{-11}{S+3}\right]$$

$= -4 L^{-1}\left[\frac{1}{s+1}\right] + 15 L^{-1}\left[\frac{1}{s+2}\right] - 11 L^{-1}\left[\frac{1}{s+3}\right]$

$= -4e^{-t} + 15e^{-2t} - 11e^{-3t}$

Que3:- Find the inverse Laplace Transform of $\frac{s+8}{s^2+4s+5}$.

Ans- Consider $s^2+4s+5=0$, $s=-2\pm i$. Complex roots.

$\Rightarrow \cancel{s^2+4s+5=0}$

$\Rightarrow \cancel{s^2+4s+2^2+5=2^2}$

$\Rightarrow \cancel{(s^2+4s+4)+5-4=0} \Rightarrow$

Consider,

s^2+4s+5

$= (s^2+4s+2^2)+5-2^2$

$= (s+2)^2+1.$

$\therefore L^{-1}\left[\frac{s+8}{s^2+4s+5}\right]$

$= L^{-1}\left[\frac{s+8}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] + L^{-1}\left[\frac{6}{(s+2)^2+1}\right]$

$= L^{-1}\left[\frac{s+2}{(s+2)^2+1^2}\right] + 6L^{-1}\left[\frac{1}{(s+2)^2+1^2}\right]$

$= e^{-2t}.\cos t + 6.\times e^{-2t}.\sin t$

$= e^{-2t}[\cos t + 6\sin t]$

Que4:- Find the $L^{-1}\left[\frac{2s+1}{s^2+2s+5}\right]$?

Ans:- Consider $s^2+2s+5=0$, $s=-1\pm 2i$ Complex roots.

Now $s^2+2s+5 = s^2+2s+1^2+5-1^2$

$= (s+1)^2+4 = (s+1)^2+4$

$\therefore L^{-1}\left[\frac{2s+1}{s^2+2s+5}\right] = L^{-1}\left[\frac{2s+1}{(s+1)^2+4}\right] = L^{-1}\left[\frac{2s+2+1-2}{(s+1)^2+4}\right]$

$= \cancel{L^{-1}\left[\frac{s+s+1}{(s+1)^2+4}\right]} = \cancel{L^{-1}[\ s\ ]}$

$= L^{-1}\left[\frac{2(s+1)-1}{(s+1)^2+4}\right] = 2.L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + L^{-1}\left[\frac{-1}{(s+1)^2+2^2}\right]$

$= 2.e^{-t}.\cos 2t - \frac{1}{2}.e^{-t}.\sin 2t$

$= e^{-t}\left[2\cos 2t - \frac{1}{2}\sin 2t\right]$

Que 5: Find $L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right]$ using partial fraction?

Ans: $L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right]$?

Consider $\frac{s^2+9s+2}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$\Rightarrow \frac{s^2+9s+2}{(s-1)^2(s+2)} = \frac{A(s-1)^2(s+2)+B(s-1)(s+2)+C(s-1)(s-1)^2}{(s-1)(s-1)^2(s+2)}$

$\Rightarrow s^2+9s+2 = \frac{(s-1)\left[A(s-1)(s+2)+B(s+2)+C(s-1)^2\right]}{(s-1)}$

$\Rightarrow s^2+9s+2 = A(s-1)(s+2)+B(s+2)+C(s-1)^2$

Put $s=1$

$1+9+2 = A\times 0 + B(1+2) + C\times 0$

$12 = 3B \Rightarrow B = 4$

Put $s=-2$

$(-2)^2+9\times(-2)+2 = A\times 0 + B\times 0 + C(-2-1)^2$

$4-18+2 = C(-3)^2$

$-12 = 9C \Rightarrow C = -12/9 \Rightarrow \underline{\underline{C = -4/3}}$

Put $s=0$

$0+0+2 = A(0-1)(0+2)+B(0+2)+C(0-1)^2$

$2 = A(-1)(2)+B(2)+C(-1)^2$

$2 = -2A+2B+C \Rightarrow 2 = -2A+(2\times 4)+\cancel{C}-4/3$

$2+\frac{4}{3} = -2A+8 \rightarrow \frac{10}{3}-8 = -2A$

$-2A = -\frac{14}{3} \Rightarrow \underline{\underline{A = 7/3}}$

$\therefore L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right] = L^{-1}\left[\frac{7/3}{s-1} + \frac{4}{(s-1)^2} + \frac{-4/3}{s+2}\right]$

$$= 7/3\, L^{-1}\left[\frac{1}{S-1}\right] + 4\, L^{-1}\left[\frac{1}{(S-1)^2}\right] + -4/3\, L^{-1}\left[\frac{1}{S+2}\right]$$

$$= 7/3 \cdot e^{+t} + 4\left[\frac{t^1}{(1-1)!}\right]_{S \to S-1} - 4/3\, e^{-2t}$$

$$= 7/3\, e^t + 4 e^t \cdot \frac{t}{1!} - 4/3\, e^{-2t}$$

$$= 7/3\, e^t + 4t e^t - 4/3\, e^{-2t}$$

Solution of Ordinary Differential Equations Using Laplace Transformation

Results:

1) $L[y'(t)] = S \cdot L[y(t)] - y(0)$
2) $L[y''(t)] = S^2 L[y(t)] - S y(0) - y'(0)$
3) $L[y'''(t)] = S^3 L[y(t)] - S^2 y(0) - S y'(0) - y''(0)$

In general

$$L[y^n(t)] = S^n L[y(t)] - S^{n-1} y(0) - S^{n-2} y'(0) - S^{n-3} y''(0) - \cdots - y^{n-1}(0)$$

Que1: Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, given $y(0) = 0$ and $y'(0) = 1$ using Laplace transformation.

Ans: Given ODE: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

$$\Rightarrow y'' - 5y' + 6y = 0.$$

Taking Laplace transform of both sides,

$$L[y'' - 5y' + 6y] = L[0]$$

$$\Rightarrow L[y''] - 5L[y'] + 6L[y] = 0$$

$$\Rightarrow S L[y] - y(0) - 5\left[S^2 L[y] - S y(0) - y'(0)\right] + 6L[y] = 0$$

$\Rightarrow s^2 L[y] - s\,y(0) - y'(0) - 5sL(y) + 5y(0) + 6L(y) = 0$

Given $y(0)=0$ and $y'(0)=1$

$\Rightarrow s^2 L[y] - s\times 0 - 1 - 5sL[y] + 5\times 0 + 6\times L[y] = 0.$

$\Rightarrow s^2 L[y] - 5sL[y] + 6L[y] = 0+1$

$\Rightarrow L[y]\left[s^2 - 5s + 6\right] = 1$

$\Rightarrow L[y] = \frac{1}{s^2-5s+6} \Rightarrow y = L^{-1}\left[\frac{1}{s^2-5s+6}\right].$

Now consider $s^2-5s+6 = (s-2)(s-3)$. be the factors.

$\therefore \frac{1}{s^2-5s+6} = \frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$

$\Rightarrow \frac{1}{(s-2)(s-3)} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)}$

$\Rightarrow 1 = A(s-3) + B(s-2)$

Put $s=3$, $1 = 0 + B(3-2) \Rightarrow \underline{B=1}$.

Put $s=2$, $1 = A(2-3) + B\times 0 \Rightarrow \underline{A=-1}$.

$\therefore y = L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{-1}{s-2}\right] + L^{-1}\left[\frac{1}{s-3}\right]$

$= -e^{2x} + e^{3x} = \underline{\underline{e^{3x} - e^{2x}}}.$

Que2:- Solve $(D^2+3D+2)y = e^{-t}$ given $y(0)=0$, $y'(0)=0$.
Using Laplace Transformation:-

Ans:- Given $(D^2+3D+2)y = e^{-t}$

$\Rightarrow D^2y + 3Dy + 2y = e^{-t} \Rightarrow y'' + 3y' + 2y = e^{-t}.$

Taking laplace transformation:-

$L[y''+3y'+2y] = L[e^{-t}]$

$\Rightarrow L[y''] + 3L[y'] + 2L[y] = L[e^{-t}]$

$\Rightarrow s^2L[y] - s\,y(0) - y'(0) + 3[sL[y] - y(0)] + 2L[y] = L[e^{-t}]$

$$\Rightarrow s^2 L[y] - 0 - 0 + 3[sL[y] - 0] + 2L[y] = \frac{1}{s+1}$$

$$\Rightarrow s^2 L[y] + 3sL[y] + 2L[y] = \frac{1}{s+1}$$

$$\Rightarrow L[y][s^2 + 3s + 2] = \frac{1}{s+1}$$

$$L[y] = \frac{1}{(s+1)(s^2+3s+2)} = \frac{1}{(s+1)(s+1)(s+2)} = \frac{1}{(s+1)^2(s+2)}$$

$$\therefore y = L^{-1}\left[\frac{1}{(s+1)^2(s+2)}\right]$$

Now, consider $\frac{1}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$.

$$\Rightarrow \frac{1}{(s+1)^2(s+2)} = \frac{A(s+1)^2(s+2) + B(s+1)(s+2) + C(s+1)(s+1)^2}{(s+1)(s+1)^2(s+2)}$$

$$\Rightarrow 1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2.$$

Put $s = -1$

$$1 = A \times 0 + B(-1+2) + C(0) \Rightarrow \underline{\underline{B = 1}}.$$

Put $s = -2$.

$$1 = A \times 0 + B \times 0 + C(-2+1)^2 \Rightarrow \underline{\underline{C = 1}}.$$

Put $s = 0$

$$1 = A(1)(2) + B(2) + C(1)^2$$

$$1 = 2A + 2B + C \Rightarrow 1 = 2A + 2 + 1$$

$$2A = 1 - 3 = -2 \Rightarrow \underline{\underline{A = -1}}$$

$$\therefore y = L^{-1}\left[\frac{1}{(s+1)^2(s+2)}\right] = L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$= -e^{-t} + \left[\frac{t^{2-1}}{(2-1)!}\right]_{s \to s+1} + e^{-2t}.$$

$$= -e^{-t} + \frac{t}{1!} \cdot e^{-t} + e^{-2t}$$

$$= \underline{\underline{-e^{-t} + te^{-t} + 2e^{-t}}}$$

Que3:- Using L.T solve $y''+5y'+6y=e^{-t}$ given $y(0)=0, y'(0)=1$

Ans:- Given $y''+5y'+6y=e^{-t}$.

Taking L.T on both sides;

$$L[y''+5y'+6y]=L[e^{-t}].$$

$$\Rightarrow L[y'']+5L[y']+6L[y]=L[e^{-t}]$$

$$\Rightarrow s^2 L[y]-s\,y(0)-y'(0)+5\left[s\,L[y]-y(0)\right]+6L[y]=L[e^{-t}]$$

$$\Rightarrow s^2L[y]-0-1+5[sL[y]-0]+6L[y]=\frac{1}{s+1}$$

$$\Rightarrow s^2L[y]-1+5sL[y]+6L[y]=\frac{1}{s+1}$$

$$\Rightarrow L[y]\left[s^2+5s+6\right]=\frac{1}{s+1}+1=\frac{1+s+1}{s+1}=\frac{s+2}{s+1}$$

$$\Rightarrow L[y]=\frac{s+2}{(s+1)(s^2+5s+6)} \Rightarrow L[y]=\frac{s+2}{(s+1)(s+2)(s+3)}$$

$$\Rightarrow L[y]=\frac{1}{(s+1)(s+3)} \Rightarrow y=L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]$$

Consider $\frac{1}{(s+1)(s+3)}=\frac{A}{s+1}+\frac{B}{s+3}$

$$\Rightarrow \frac{1}{(s+1)(s+3)}=\frac{A(s+3)+B(s+1)}{(s+1)(s+3)}$$

$$\Rightarrow 1=A(s+3)+B(s+1)$$

Put $s=-3$, $1=A\times 0+B(-3+1)$

$$\Rightarrow 1=0+B(-2)=-2B=1 \Rightarrow \underline{\underline{B=-1/2}}.$$

Put $s=-1$, $1=A(-1+3)+B(-1+1)$

$$1=A(2)\Rightarrow A=\underline{1/2}$$

$$y=L^{-1}\left(\frac{1}{(s+1)(s+3)}\right)=L^{-1}\left[\frac{1/2}{s+1}\right]+L^{-1}\left[\frac{-1/2}{s+3}\right].$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}$$

$$=\underline{\underline{\frac{e^{-t}-e^{-3t}}{2}}}.$$

Quest:- Solve using the method of laplace Transformation:-

Ans:- Given $y''+2y'+y=e^{-t}$, $y(0)=0$, $y'(0)=1$.

Taking L.T on both sides:-

$$L[y'']+2L[y']+L[y]=L[e^{-t}]$$

$$\Rightarrow s^2L[y]-sy(0)-y'(0)+2[sL[y]-y(0)]+L[y]=L[e^{-t}]$$

$$\Rightarrow s^2L[y]-0-1+2[sL[y]-0]+L[y]=\frac{1}{s+1}$$

$$\Rightarrow s^2L[y]-1+2sL[y]+L[y]=\frac{1}{s+1}$$

$$\Rightarrow L[y](s^2+2s+1)=\frac{1}{s+1}+1$$

$$\Rightarrow L[y](s^2+2s+1)=\frac{1+s+1}{s+1}=\frac{s+2}{s+1}$$

$$\Rightarrow L[y]=\frac{s+2}{(s+1)(s^2+2s+1)} \Rightarrow y=L^{-1}\left[\frac{s+2}{(s+1)(s+1)(s+1)}\right]$$

$$\Rightarrow y=L^{-1}\left[\frac{s+2}{(s+1)^3}\right]$$

Consider $\frac{s+2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$.

$$\Rightarrow \frac{s+2}{(s+1)^3}=\frac{A(s+1)^2(s+1)^3+B(s+1)(s+1)^3+C(s+1)(s+1)^2}{(s+1)(s+1)^2(s+1)^3}$$

$$\Rightarrow s+2=\frac{(s+1)^3[A(s+1)^2+B(s+1)+C]}{(s+1)(s+2)^2}$$

$$\Rightarrow s+2=A(s+1)^2+B(s+1)+C.$$

Put $s=0$, $2=A(0+1)^2+B(0+1)+C$.

$$\Rightarrow 2=A+B+C \Rightarrow A+B+C=2 \quad \text{--- (1)}$$

Put $s=-1$, $-1+2=A(0)+B(0)+C$.

$$\Rightarrow 1=C \Rightarrow \underline{\underline{C=1}}$$

$$s+2=A(s^2+2s+1)+B(s+1)+C$$

$$\Rightarrow s+2=As^2+2As+A+Bs+B+C.$$

$$\Rightarrow s+2=As^2+(2A+B)s+(A+B+C)$$

Equating the coefficients of s^2, $A=0$.

$\Rightarrow$ from (1) $A+B+C=2$

$\Rightarrow 0+B+1=2 \Rightarrow \underline{\underline{B=1}}$

$$\therefore y = L^{-1}\left[\frac{s+2}{(s+1)^2}\right] = L^{-1}\left[0+\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)^2}\right] + L^{-1}\left[\frac{1}{(s+1)^3}\right]$$

$$= \left[\frac{t^{2-1}}{(2-1)!}\right]_{s\to s+1} + \left[\frac{t^{3-1}}{(3-1)!}\right]_{s\to s+1}$$

$$= e^{-t}\cdot\frac{t}{1!} + e^{-t}\cdot\frac{t^2}{2!} \Rightarrow \underline{\underline{y = te^{-t} + \frac{t^2e^{-t}}{2}}}.$$

Que 5:- Solve $y''+4y'+3y=e^{-t}$, $y(0)=1$, $y'(0)=1$, using L.T.

Ans:- Given $y''+4y'+3y=e^{-t}$.

Taking L.T $\Rightarrow L[y'']+4L[y']+3L[y]=L[e^{-t}]$

$\Rightarrow s^2L[y]-sy(0)-y'(0)+4[sL(y)-y(0)]+3L[y]=L[e^{-t}]$

$\Rightarrow s^2L[y]-(s\times1)-1+4[sL(y)-1]+3L[y]=L[e^{-t}]$

$\Rightarrow s^2L[y]-s-1+4sL[y]-4+3L[y]=L[e^{-t}]$

$$\Rightarrow L[y][s^2+4s+3]-s-5=\frac{1}{s+1}$$

$$\Rightarrow L[y][s^2+4s+3]=\frac{1}{s+1}+(s+5)=\frac{1+(s+5)(s+1)}{s+1}$$

$$\Rightarrow L[y]=\frac{1+s^2+s+5s+5}{(s+1)(s^2+4s+3)}=\frac{s^2+6s+6}{(s+1)(s+1)(s+3)}$$

$$\Rightarrow y=L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right]$$

Now, consider, $\dfrac{s^2+6s+6}{(s+1)^2(s+3)}=\dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s+3}$

$$\Rightarrow \frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A(s+1)^2(s+3)+B(s+1)(s+3)+C(s+1)(s+1)^2}{(s+1)(s+1)^2(s+3)}$$

$$\Rightarrow \frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{(s+1)\left[A(s+1)(s+3)+B(s+3)+C(s+1)^2\right]}{(s+1)(s+1)^2(s+3)}$$

$$\Rightarrow s^2+6s+6 = A(s+1)(s+3)+B(s+3)+C(s+1)^2$$

put $s=-1$, $1-6+6 = A\times 0 + B(-1+3)+C(0)$

$$\Rightarrow 1 = B(2) \Rightarrow \underline{\underline{B = 1/2}}.$$

put $s=-3$, $9-18+6 = A(0)+B(0)+C(-3+1)^2$

$$\Rightarrow -3 = C(-2)^2 \Rightarrow -3 = 4C \Rightarrow C = -3/4.$$

put $s=0$, $6 = A(1)(3)+B(3)+C(1)^2$

$$\Rightarrow 6 = 3A+3B+C.$$

$$\Rightarrow 6 = 3A + 3\times\frac{1}{2} - \frac{3}{4}$$

$$\Rightarrow 6 = 3A + \frac{3}{4}$$

$$\frac{2\times 3}{2\times 2} - \frac{3}{4} = \frac{6-3}{4}$$

$$\Rightarrow 3A = 6-\frac{3}{4} = \frac{21}{4}$$

$$\Rightarrow A = 7/4.$$

$$\therefore y = L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] = L^{-1}\left[\frac{7/4}{s+1}\right] + L^{-1}\left[\frac{1/2}{(s+1)^2}\right] + L^{-1}\left[\frac{-3/4}{s+3}\right]$$

$$= \underline{\underline{7/4\, e^{-t} + 1/2\, e^{-t}.t - 3/4\, e^{-3t}}}.$$

Convolution Theorem

Let $f(t)$ and $g(t)$ be two functions, then the convolution of $f(t)$ and $g(t)$ is defined as,

$$\boxed{f(t) * g(t) = \int_0^t f(u).g(t-u)\,du}$$

Now, let $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

$$\Rightarrow f(t) = L^{-1}[F(s)] \text{ and } g(t) = L^{-1}[G(s)]$$

$$\Rightarrow \boxed{L^{-1}[F(s)] * L^{-1}[G(s)] = \int_0^t f(u).g(t-u)\,du}$$

Note :-

1. If $f(t)$ and $g(t)$ be two functions, then

$$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$$

$$\Rightarrow L[f(t) * g(t)] = F(s) \cdot G(s).$$

2. $$L^{-1}[F(s) \cdot G(s)] = f(t) * g(t).$$

$$\Rightarrow \boxed{L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]}$$

Que 1 :- Find $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$ using convolution theorem.

(7)

Ans :- $$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = L^{-1}\left[\frac{1}{s+1} \cdot \frac{1}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t} * \sin t \qquad f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

$$\Rightarrow e^{-t} * \sin t = \int_0^t e^{-u} \sin(t-u)\, du.$$

$$\int e^{at} \sin bt\, dt = \frac{e^{-at}}{a^2+b^2}\left[a \sin bt - b \cos bt\right]$$

$$= \left[\frac{e^{-u}}{(-1)^2+(-1)^2}\left[(-1)\sin(t-u) - (-1)\cos(t-u)\right]\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[0+1\right] - \frac{1}{2}\left[(-1)\sin t + \cos t\right]$$

$$= \frac{e^{-t}}{2} - \frac{1}{2}\left[\cos t - \sin t\right]$$

$$= \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

Que 2:- Using convolution theorem, find the inverse L.T of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Ans:- $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$

$= \cos at * \cos bt$

We know that $f(t) * g(t) = \int_0^t f(u).g(t-u)\,du$.

$\Rightarrow \cos at * \cos bt = \int_0^t \cos au . \cos b(t-u)\,du$.

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$= \int_0^t \frac{1}{2}\left[\cos(au+b(t-u)) + \cos(au-b(t-u))\right] du.$

$= \int_0^t \frac{1}{2}\left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du.$

$= \int_0^t 1/2\left[\cos((a-b)u+bt) + \cos((a+b)u-bt)\right] du$

$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$

$= \frac{1}{2}\left\{\left[\frac{\sin[(a-b)t+bt]}{a-b} + \frac{\sin[(a+b)t-bt]}{a+b}\right] - \left[\frac{\sin[0+bt]}{a-b} + \frac{\sin[0-bt]}{a+b}\right]\right\}$

$= \frac{1}{2}\left\{\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right\}$

$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right\}$

$= \frac{1}{2}\left[\frac{a\sin at - a\sin bt + b\sin at - b\sin bt + a\sin at + a\sin bt - b\sin at - b\sin bt}{(a-b)(a+b)}\right]$

$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$

Que 38:- Using convolution theorem find $L^{-1}\left[\frac{2}{(s^2+1)(s^2+25)}\right]$

Ans:- $L^{-1}\left[\frac{2}{(s^2+1)(s^2+25)}\right] = 2 \cdot L^{-1}\left[\frac{1}{(s^2+1)(s^2+25)}\right]$

$= 2 \cdot L^{-1}\left[\frac{1}{s^2+1}\right] * L^{-1}\left[\frac{1}{s^2+25}\right]$

$= 2 \cdot \left[\frac{1}{1} \cdot \sin t * \frac{1}{5} \cdot \sin 5t\right]$

We know that $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$.

$\Rightarrow L^{-1}\left[\frac{2}{(s^2+1)(s^2+25)}\right] = 2 \cdot \int_0^t \sin u \cdot \frac{1}{5} \cdot \sin 5(t-u)\, du$.

$= \frac{2}{5} \int_0^t \sin u \cdot \sin 5(t-u)\, du$.

$\sin A \cdot \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$= \frac{2}{5} \int_0^t \frac{1}{2}\left[\cos(u - 5(t-u)) - \cos(u + 5(t-u))\right] du$

$= \frac{1}{5} \int_0^t \left[\cos(u - 5t + 5u) - \cos(u + 5t - 5u)\right] du$

$= \frac{1}{5} \int_0^t \left[\cos(-5t + 6u) - \cos(5t - 4u)\right] du$

$= \frac{1}{5} \int_0^t \left[\cos(6u - 5t) - \cos(5t - 4u)\right] du$

$= \frac{1}{5}\left[\frac{\sin(6u-5t)}{6} - \frac{\sin(5t-4u)}{-4}\right]_0^t$

$= \frac{1}{5}\left[\left(\frac{\sin(6t-5t)}{6} - \frac{\sin(5t-4t)}{-4}\right) - \left(\frac{\sin(0-5t)}{6} - \frac{\sin(5t)}{-4}\right)\right]$

$= \frac{1}{5}\left[\frac{\sin t}{6} + \frac{\sin t}{4} + \frac{\sin 5t}{6} - \frac{\sin 5t}{4}\right]$

$= \frac{1}{5}\left[\frac{10 \sin t}{24} - \frac{2 \sin 5t}{24}\right]$

$\frac{\sin(6u-t)}{6} - \frac{\sin(4u+t)}{4}$

$\frac{\sin(5t)}{6} - \frac{\sin(4t)}{4}$

$= \frac{1}{5}\left[\frac{5 \sin t}{12} - \frac{\sin 5t}{12}\right]$

$= \frac{\sin t - \sin 5t}{12}$

Que 4:- Using convolution theorem, find $L^{-1}\left[\frac{18s}{(s^2+36)^2}\right]$

Ans:- $L^{-1}\left[\frac{18s}{(s^2+36)^2}\right] = L^{-1}\left[\frac{18}{s^2+36} \cdot \frac{s}{s^2+36}\right]$

$= L^{-1}\left[\frac{18}{s^2+36}\right] * L^{-1}\left[\frac{s}{s^2+36}\right]$

$= 18 \times \frac{1}{6} \sin 6t * \frac{1}{6} \cdot \cos 6t$

$= 3 \sin 6t * \frac{1}{6} \cos 6t$

$= \int_0^t 3 \cdot \sin 6u \cdot \frac{1}{6} \cos(6(t-u))\, du$

$= \frac{3}{6} \int_0^t \sin 6u \cdot \cos(6t - 6u)\, du$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$= \frac{1}{2} \int_0^t \frac{1}{2}\left[\sin(6u+6t-6u) + \sin(6u-(6t-6u))\right] du$

$= \frac{1}{4} \int_0^t \left[\sin(6t) + \sin(12u - 6t)\right] du$

$= \frac{1}{4} \left[\sin 6t \cdot u + \frac{\sin(12u-6t)}{12}\right]_0^t$

$= \frac{1}{4}\left[\left(t \sin 6t + \frac{\sin(6t)}{12}\right) - \left(0 + \frac{\sin(-6t)}{12}\right)\right]$

$= \frac{1}{4}\left[t \sin 6t + \frac{\sin 6t}{12} + \frac{\sin 6t}{12}\right]$

$= \frac{1}{4}\left[t \sin 6t + \frac{2 \sin 6t}{12}\right] = \frac{1}{4}\left[t \sin 6t + \frac{\sin 6t}{6}\right]$

$= \frac{t \sin 6t}{4} + \frac{\sin 6t}{24} = \frac{6t \sin 6t + \sin 6t}{24}$

Que 5:- Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ using Convolution Theorem.

Ans:- $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at * \cos at.$

We know that $\int_0^t f(u).g(t-u)\,du = f(t) * g(t)$

$\Rightarrow \cos at * \cos at = \int_0^t \cos au.\cos a(t-u)\,du.$

$= \int_0^t \cos au.\cos(at-au)\,du.$

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$= \int_0^t \frac{1}{2}\left[\cos(au+at+au) + \cos(au-(at-au))\right]du$

$= \frac{1}{2}\int_0^t \left[\cos(at) + \cos(2au-at)\right]du.$

$= \cancel{\frac{1}{2}\left[\frac{\sin at}{a} + \sin(2au-\right.}$

$= \frac{1}{2}\left[\cos at.u + \frac{\sin(2au-at)}{2a}\right]_0^t$

$= \frac{1}{2}\left[t\cos at + \frac{\sin(2at-at)}{2a}\right] - \left[0 + \frac{\sin(-at)}{2a}\right]$

$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$

$= \frac{1}{2}\left[t\cos at + \frac{2\sin at}{2a}\right] = \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$

$= \frac{1}{2}\left[\frac{at\cos at + \sin at}{a}\right]$

$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[\frac{at\cos at + \sin at}{a}\right]$

Que 6:- Using Convolution theorem find Inverse Laplace Transform $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

Ans:- $L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}.\frac{1}{s^2+1}\right]$

$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+1}\right] = 1 * \sin t.$

$= \int_0^t 1.\sin(t-u)\,du = \left[\frac{\cos(t-u)}{-1}\right]_0^t = \frac{-\cos 0 - -\cos(-t)}{-1}$

$= \frac{-1+\cos t}{-1} = \underline{\underline{\cos t - 1}}.$