MODULE : 3

Ordinary differential equations (ODE)

Homogenous | non-homogenous

Homogenous linear ode with constant coefficients: the general form of a 2nd order homogeneous linear ode with constant coefficient is given by;

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

OR

$$y'' + ay' + by = 0$$

Egs -

$$y'' + 3y' + 2y = 0$$

The General solution of homogeneous linear ode:

If y_1 and y_2 are two solutions of a second order homogeneous linear ode then the general solution is given by;

$$\boxed{y = c_1y_1 + c_2y_2}$$

Linear Independent and linear dependent

Two solutions y_1 and y_2 are said to be linearly dependent if $\frac{y_1}{y_2}$ = a constant otherwise linearly independent

<u>Note</u>: The $\{y_1, y_2\}$ of linearly independent solutions is called a basis

1. Check whether $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent or not.

A: consider $\frac{y_1}{y_2} = \frac{e^x}{e^{-x}} = e^x \cdot e^x$

$= e^{x+x} = e^{2x} \neq$ constant.

$\therefore$ y_1 and y_2 are linearly independent

2. Check whether the following are linearly independent or not

① $y_1 = \sin x$; $y_2 = \cos x$

$\frac{y_1}{y_2} = \frac{\sin x}{\cos x} = \tan x$

$\therefore$ independent

② $y_1 = x^2$, $y_2 = 1/x$

$\frac{y_1}{y_2} = \frac{x^2}{1/x} = x^2 \cdot x$

$= x^3$

$\therefore$ independent

③ $y_1 = e^{2x}$, $y_2 = e^{2x}$

$\frac{y_1}{y_2} = \frac{e^{2x}}{e^{2x}} = \cancel{e^{2x}} \cdot e = 1$

$\therefore$ dependent

3. Check whether $y_1 = x^2, y_2 = 2x$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & 2x \\ 2x & 2 \end{vmatrix}$$

$$= 2x^2 - 4x^2 \neq 0 \quad = -2x^2$$

$\therefore$ linearly independent

Solution of a second order homogeneous linear ode with constant coefficient.

A second order homogeneous ode is given by $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ $\boxed{\frac{d}{dx} = D}$

put $\frac{d^2y}{dx^2} = D^2y$ and $\frac{dy}{dx} = Dy$

$\therefore$ the above equation becomes

$\Rightarrow D^2y + aDy + by = 0$

$(D^2 + aD + b)y = 0$ auxilary

Auxilary equation is — replacing $D \to m$ D by m, $D^2 = m^2$, neglet y

$\Rightarrow m^2 + am + b = 0$

which is a quadratic equation and find the values of m

Let $m = m_1$ & m_2

$\therefore$ the solution is given by

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

where c_1 and c_2 are constants

same

$y = (c_1 + c_2 x) e^{mx}$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

type 1: [roots are different]

Q) solve $y'' + 5y' + 6y = 0$

given equation is ⇒

$$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 6y = 0$$

$$\boxed{\frac{d^2y}{dx^2} = D^2y \quad \frac{dy}{dx} = Dy}$$

$$D^2y + 5Dy + 6y = 0$$

$$(D^2 + 5D + 6)y = 0$$

AE ⇒ $m^2 + 5m + 6 = 0$

Let $m_1 = -2$, $m_2 = -3$

roots are different

∴ general soln is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$= c_1 e^{-2x} + c_2 e^{-3x}$$

2. Find a basis of solutions and general solution for ~~$y'' + 5y' + 6y = 0$~~ $y'' - 3y' + 2y = 0$

A : given equation is ⇒

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0$$

$$D^2y - 3Dy + 2y = 0$$

$$(D^2 - 3D + 2)y = 0$$

AE ⇒ $m^2 - 3m + 2 = 0$

$m_1 = 2$, $m_2 = 1$

roots are different

∴ general soln is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$= c_1 e^{2x} + c_2 e^{x}$$

roots are different

$\therefore$ general soln is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

type 2: [roots are real and equal]

Let $m = m_1 = m_2$

$\therefore$ the general solution is $\Rightarrow$

$$\boxed{y = (c_1 + c_2 x)\, e^{mx}}$$

1. Find the general solution for $y'' + 6y' + 9y = 0$

given equation $\Rightarrow$ $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

$D^2y + 6Dy + 9y = 0$

$(D^2 + 6D + 9)y = 0$

AE $\Rightarrow$ $m^2 + 6m + 9 = 0$

$m = -3, -3$ (roots are same)

general soln $\Rightarrow$ $y = (c_1 + c_2 x)\, e^{mx}$

$= (c_1 + c_2 x)\, e^{-3x}$

2. Find the general soln of $y'' + y' + 0.25y = 0$

given $\Rightarrow$ $\frac{d^2y}{dx} + \frac{dy}{dx} + 0.25y = 0$

$D^2y + Dy + 0.25y = 0$

$(D^2 + D + 0.25)y = 0$

$AE \Rightarrow \quad m^2 + m + 0.25 = 0$

$m = \frac{-1}{2}, \frac{-1}{2} = -0.5$

$\therefore$ general soln $\Rightarrow y = (C_1 + C_2 x) e^{mx}$

$= (C_1 + C_2 x) e^{-0.5x}$

type 3 : [Complex root]

Let $m = \alpha \pm i\beta$

$\therefore$ general solution is $\Rightarrow$

$$\boxed{y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]}$$

α = Real part

β = imaginary part

1. Solve $y'' + 4y' + 29y = 0$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 29y = 0$$

$D^2y + 4Dy + 29y = 0$

$(D^2 + 4D + 29)y = 0$

$AE \Rightarrow \quad m^2 + 4m + 29 = 0$ (Complex root)

~~AE~~ $\Rightarrow$ $m_1 = -2 + 5i$ $\quad m_2 = -2 - 5i$

$m = -2 \pm 5i, \quad \alpha = -2, \quad \beta = 5$

$\therefore$ general soln $= e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

$= e^{-2x} [c_1 \cos 5x + c_2 \sin 5x]$

2. Solve $y'' + 4y = 0$

$$\frac{d^2y}{dx} + 4_y \frac{dy}{dx} = 0$$

$$D^2y + 4Dy = 0$$

$$(D^2 + 4D)y = 0$$

AE $\Rightarrow m^2 + 0 + 4 = 0$

$m = 0 \pm 2i$ [complex root]

$\alpha = 0 \quad \beta = 2$

general soln

$$= e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

$$= e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$$

$$= c_1 \cos 2x + c_2 \sin 2x$$

1. solve initial value problem (IVP)

$y'' + y' - 6y = 0, \quad y(0) = 10, \quad y'(0) = 0$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \qquad x = 0, \; y = 10 \qquad x = 0, \; y' = 0$$

$$D^2y + Dy - 6y = 0$$

$$(D^2 + D - 6)y = 0$$

AE $= m^2 + m - 6 = 0$

$m_1 = 2, \quad m_2 = -3$

(roots are diff)

general soln ⇒

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{2x} + c_2 e^{-3x} \quad \text{——— (1)}$$

(1) given that $y(0) = 10$

ie, when $x = 0$, $y = 10$

sub in equ (1) ⇒

$$10 = c_1 e^{2 \cdot 0} + c_2 e^{-3 \cdot 0}$$

$$10 = c_1 e^0 + c_2 e^0$$

$$10 = c_1 + c_2 \quad \text{——— (2)}$$

differentiating equ (1) w.r.t. x

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

$$y' = c_1 2 \cdot e^{2x} + c_2 - 3 e^{-3x} \quad \text{——— (3)}$$

(2) given that $y'(0) = 0$

ie, when $x = 0$, $y' = 0$

∴ the above equation becomes ⇒

$$0 = 2 c_1 e^0 - 3 c_2 e^0$$

$$0 = 2 c_1 - 3 c_2 \quad \text{——— (4)}$$

solve equ (2) and (4) ⇒

$$c_1 = 6$$

$$c_2 = 4$$

sub in equ (1)

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

$$= 6 e^{2x} + 4 e^{-3x}$$

... of solutions are given :

1. Find a second order homogeneous linear ode for which the functions are the solutions e^{-x}, e^{-2x}

A: Let $y_1 = e^{-x}$ and $y_2 = e^{-2x}$

we know that general soln is,

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

sub. y_1 & $y_2 \Rightarrow$

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

this is type I ie,

$m_1 = -1$ $m_2 = -2$ [roots are diff.]

ie, $(D-m_1)(D-m_2) = 0$

$(D+1)(D+2) = 0$

$\Rightarrow D^2 + 2D + D + 2 = 0$

$D^2 + 3D + 2 = 0$

$\therefore$ the equation is $(D^2+3D+2)y = 0$

$$D^2y + 3Dy + 2y = 0$$

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = 0$$

2. Find 2nd order homogenous equation if the solns are $\cos 5x$ and $\sin 5x$

let $y_1 = \cos 5x$, $y_2 = \sin 5x$

gen soln $y = c_1y_1 + c_2y_2$

$\Rightarrow y = c_1 \cos 5x + c_2 \sin 5x$ [type 3:]

$[\because y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)]$

$\alpha = 0$, $\beta = 5$

Roots are $\alpha \pm i\beta$

$0 \pm i5$

$\pm 5i$

$m_1 = 5i \qquad m_2 = -5i$

$(D-m_1)(D-m_2) = 0$

$\underbrace{(D-5i)}_{(a-b)} \; \underbrace{(D+5i)}_{(a+b)} = 0 \qquad = a^2 - b^2$

Here $a = D$
$b = 5i$

$D^2 - (5i)^2 = 0$

$D^2 + 5^2 = 0$

$D^2 + 25 = 0$

eqn $\Rightarrow$ $D^2 y + 25y = 0$

ie $\dfrac{d^2y}{dx^2} + 25y = 0$

Solution of non-homogeneous linear ode with constant coefficient :

The general soln of non-homogeneous differen-tial equation is given by ;

$$\boxed{y = y_h + y_p}$$

where y_h is the complementory Function (CF)

y_p is particular Integral (PI)

To find y_h, convert the given non-homogeneous equation into homogeneous and find the soln.

The table given below is to find y_p :-

RHS	y_p
ke^{ax}	Ae^{ax}
$k\cos ax$ OR $k\sin ax$	$A\cos ax + B\sin ax$
kx^n	$A + Bx + Cx^2 + \cdots$
$ke^{ax}\cdot\cos bx$ OR $ke^{ax}\cdot\sin bx$	$e^{ax}\left[A\cos ax + B\sin ax\right]$

1. Solve $y'' + 5y' + 6y = 2e^{-x}$

Ae^{ax}

given equation is a non-homogeneous equation convert it into homogeneous

$\Rightarrow y'' + 5y' + 6y = 0$

$D^2y + 5Dy + 6y = 0$

$$(D^2+5D+6)y = 0$$

$AE \Rightarrow \quad m^2 + 5m + 6 = 0$

$m_1 = -2 \qquad m_2 = -3 \qquad$ [diff roots]

$$\therefore y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{-2x} + c_2 e^{-3x} \quad \text{——} \ (1)$$

From the table $\quad y_p = Ae^{ax} \quad [2e^{-x}]$

$k=2$
$a=-1$

here a is -1

$$\therefore y_p = Ae^{-x} \quad \text{——} \ (2)$$

Differentiating equ (2) w.r.to x

$$y'_p = Ae^{-x}(-1)$$

$$y''_p = Ae^{-x}\cdot 1$$

$$\boxed{y = y_h + y_p}$$

$\therefore$ given equation becomes:

$$y''_p + 5y'_p + 6y_p = 2e^{-x}$$

$$Ae^{-x} - 5Ae^{-x} + 6Ae^{-x} = 2e^{-x}$$

find values of $A \Rightarrow$

$$\cancel{2}A\cancel{e^{-x}} = \cancel{2}\cancel{e^{-x}}$$

$$\therefore A = 1$$

sub in equ (2)

$$y_p = e^{-x} \quad \text{——} \ (3)$$

$$y = y_h + y_p$$

$$= c_1 e^{-2x} + c_2 e^{-x} + c_1 e^{-3x}$$

3. Find the particular integral of $y'' - 4y = \underbrace{\sin 2x}_{y_p}$

$\cancel{D^2y - 4y = \sin}$ $\qquad y''_p - 4y_p$

$y'' - 4y = \sin 2x$

$y_p = A \cos \underset{2}{ax} + B \sin \overset{2}{ax}$

Here $a = 2$

$\therefore y_p = A \cos 2x + B \sin 2x$ —— ①

$y'_p = -A\, 2 \sin 2x + B\, 2 \cos 2x$

$y_p'' = -A\, 2 \cos 2x \cdot 2 + -B\, 2 \sin 2x \cdot 2$

$= -4A \cos 2x - 4B \sin 2x$

given eqn is

$y_p'' - 4y_p = \sin 2x$

$(-4A \cos 2x - 4B \sin 2x) - 4(A \cos 2x + B \sin 2x) = \sin 2x$

$\cancel{-4A \cos 2x - 4B \sin 2x - 4A \cos 2x - 4B \sin 2x = \sin 2x}$

$\cancel{\therefore \sin 2x = 0}$

$\underline{-4A \cos 2x} - 4B \sin 2x - \underline{4A \cos 2x} - 4B \sin 2x = \sin 2x$

$-8B = 1$

$-8A = 0 \qquad -8A \cos 2x - 8B \sin 2x = \sin 2x$

equating coefficients of $\cos 2x$ on both side, we get

$$-8A = 0$$

$$\Rightarrow \boxed{A = 0}$$

equating coefficients of $\sin 2x$ on both sides, we get,

$$-8B = 1$$

$$\Rightarrow \boxed{B = -\frac{1}{8}}$$

$\therefore$ equ (1) $\Rightarrow$

$$y_p = 0 \cdot \cos 2x + \left(-\frac{1}{8} \sin 2x\right)$$

$$= -\frac{1}{8} \sin 2x$$

4. solve $y'' - 4y' - 5y = 4 \cdot \cos 2x$

$y_h \Rightarrow$ homogeneous equation corresponding to the given equ is

$$y'' - 4y' - 5y = 0$$

$$\frac{d^2y}{dx} - 4\frac{dy}{dx} - 5y = 0$$

$$(D^2 - 4D - 5)y = 0$$

AE $\Rightarrow$ $m^2 - 4m - 5 = 0$

$m_1 = 5, -1$ [diff. roots]

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{5x} + c_2 e^{-x}$$

$$y_p = A \cdot \cos ax + B \cdot \sin ax$$

here, $a = 2$

$y_p = A\cos 2x + B\sin 2x$ $= 4\cos 2x$

$y_p' = -A\,2\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

$y_p'' - 4y_p' - 5y_p = 4\cos 2x$

$[-4A\cos 2x - 4B\sin 2x] - 4[-A\,2\sin 2x + 2B\cos 2x]$

$- 5[A\cos 2x + B\sin 2x] = 4\cos 2x$

$(-4A\cos 2x)(-4B\sin 2x) + (4A\sin 2x) - (8B\cos 2x)$
$(-5A\cos 2x)(-B\sin 2x) = 4\cos 2x$

$-9A\cos 2x \; \overset{-5}{\cancel{-12B}}\sin 2x + 4A\sin 2x = 4\cos 2x$
$-8B\cos 2x$

equ by $\cos 2x$

$-9A - 8B = 4$ — (2)

equ by $\sin 2x$

$-5A - 9B = 0$ — (3)

solve equ (2) & (3)

$A = \frac{-36}{145}$

$B = \frac{32}{145}$

5. Find the particular integral of $y'' - 2y' + 5y = $

$y_p = A + Bx + Cx^2$ — (1)

$y_p' = 0 + B + 2Cx$

$y_p'' = 0 + 0 + 2C \cdot 1$

$$y_p'' - 2y_p' + 5y_p = x^2$$

$$2C - 2(B+2Cx) + 5(A + Bx + Cx^2) = x^2$$

$$2C - 2B - 4Cx + 5A + 5Bx + 5Cx^2 = x^2$$

equating the coefficients of x^2, x, and constants

$\underline{x^2}$

$$5C = 1$$
$$C = 1/5$$

$\underline{x}$

$$-4C + 5B = 0$$
$$-4 \cdot \frac{1}{5} + 5B = 0$$
$$-\frac{4}{5} + 5B = 0$$
$$B = \frac{4}{25}$$

$\underline{\text{constants}}$

$$2C - 2B + 5A = 0$$
$$\left(2 \times \frac{1}{5}\right) - 2 \times \frac{4}{25} + 5 \times A = 0$$
$$\frac{2}{5} - \frac{8}{25} + 5A = 0$$
$$\frac{2}{25} + 5A = 0$$
$$A = \frac{-2}{125}$$

$$y_p = A + Bx + Cx^2 + \ldots$$
$$= \frac{-2}{125} + \frac{4}{25}x + \frac{1}{5}x^2$$