Logic and Propositional Calculus

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(*a*) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(*b*) "p or q"

p	$\neg p$
T	F
F	T

(*c*) "not p"

Fig. 4-1

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p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(*a*)

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

(*b*)

Fig. 4-2

TAUTOLOGIES AND CONTRADICTIONS

Note that the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(*a*) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(*b*) $p \wedge \neg p$

Fig. 4-5

LOGICAL EQUIVALENCE

Two propositions P(p,q,...)and Q(p,q,...)are said to be logically equivalent, or simply equivalent or equal, denoted by P(p,q, ...)≡ Q(p, q, ...) if they have identical truth tables.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

Fig. 4-6

CONDITIONAL AND BICONDITIONAL STATEMENTS

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b) $p \leftrightarrow q$

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(c) $\neg p \vee q$

Fig. 4-7

Solved Problems

PROPOSITIONS AND TRUTH TABLES

4.1. Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements: (*a*) $\neg p$; (*b*) $p \wedge q$; (*c*) $p \vee q$; (*d*) $q \vee \neg p$.

In each case, translate $\wedge$, $\vee$, and $\sim$ to read "and," "or," and "It is false that" or "not," respectively, and then simplify the English sentence.

(*a*) It is not cold. (*c*) It is cold or it is raining.

(*b*) It is cold and raining. (*d*) It is raining or it is not cold.

4.2. Find the truth table of $\neg p \wedge q$.

Construct the truth table of $\neg p \wedge q$ as in Fig. 4-9(*a*).

p	q	$\neg p$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(*a*) $\neg p \wedge q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(*b*) $p \vee \neg(p \wedge q)$

4.3. Verify that the proposition $p \vee \neg(p \wedge q)$ is a tautology.

Construct the truth table of $p \vee \neg(p \wedge q)$ as shown in Fig. 4-9(*b*). Since the truth value of $p \vee \neg(p \wedge q)$ is *T* for all values of p and q, the proposition is a tautology.

4.4. Show that the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Construct the truth tables for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ as in Fig. 4-10. Since the truth tables are the same (both propositions are false in the first case and true in the other three cases), the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent and we can write

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

4.5. Use the laws in Table 4-1 to show that $\neg(p \wedge q) \vee (\neg p \wedge q) \equiv \neg p$.

Statement		Reason
(1) $\neg(p \vee q) \vee (\neg p \wedge q)$	$\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	DeMorgan's law
(2)	$\equiv \neg p \wedge (\neg q \vee q)$	Distributive law
(3)	$\equiv \neg p \wedge T$	Complement law
(4)	$\equiv \neg p$	Identity law

CONDITIONAL STATEMENTS

4.6. Rewrite the following statements without using the conditional:

(*a*) If it is cold, he wears a hat.

(*b*) If productivity increases, then wages rise.

Recall that "If p then q" is equivalent to "Not p or q;" that is, $p \to q \equiv \neg p \vee q$. Hence,

(*a*) It is not cold or he wears a hat.

(*b*) Productivity does not increase or wages rise.

4.7. Consider the conditional proposition $p \to q$. The simple propositions $q \to p$, $\neg p \to \neg q$ and $\neg q \to \neg p$ are called, respectively, the *converse*, *inverse*, and *contrapositive* of the conditional $p \to q$. Which if any of these propositions are logically equivalent to $p \to q$?

Construct their truth tables as in Fig. 4-11. Only the contrapositive $\neg q \to \neg p$ is logically equivalent to the original conditional proposition $p \to q$.

p	q	$\neg p$	$\neg q$	Conditional $p \to q$	Converse $q \to p$	Inverse $\neg p \to \neg q$	Contrapositive $\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Fig. 4-11

4.8. Determine the contrapositive of each statement:

(*a*) If Erik is a poet, then he is poor.

(*b*) Only if Marc studies will he pass the test.

(*a*) The contrapositive of $p \to q$ is $\neg q \to \neg p$. Hence the contrapositive follows:

If Erik is not poor, then he is not a poet.

(*b*) The statement is equivalent to: "If Marc passes the test, then he studied." Thus its contrapositive is:

If Marc does not study, then he will not pass the test.

QUANTIFIERS AND PROPOSITIONAL FUNCTIONS

4.15. Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

(a) $(\exists x \in A)(x + 3 = 10)$ (c) $(\exists x \in A)(x + 3 < 5)$

(b) $(\forall x \in A)(x + 3 < 10)$ (d) $(\forall x \in A)(x + 3 \leq 7)$

(a) False. For no number in A is a solution to $x + 3 = 10$.

(b) True. For every number in A satisfies $x + 3 < 10$.

(c) True. For if $x_0 = 1$, then $x_0 + 3 < 5$, i.e., 1 is a solution.

(d) False. For if $x_0 = 5$, then $x_0 + 3$ is not less than or equal 7. In other words, 5 is not a solution to the given condition.