Module 2: Syllabus Discription (9Hrs)

Homogeneous ODES of Second order, Superposition principle, General solution. Homogeneous linear ODES of second order with constant coefficients (Method to find general solutions, Solution of linear initial valued problem). Non-Homogeneous ODES (with constant coefficient General solutions, Particular solution by functions ke^{Ax}, kx^n, $k\cos\omega x$, $k\sin\omega x$, $ke^{ax}\cdot\cos\omega x$, $ke^{ax}\cdot\sin\omega x$). Initial Value Problem for Non-Homogeneous Second order linear ODE (with Constant coefficients) solution ~~of~~ by Variation of parameters (Second order).

Ordinary Differential Equations:-

Differential Equations:- Differential equations are the equation which involves derivatives.

Eg:- $\frac{dy}{dx} + 2y = 0$ $\qquad \frac{d^2y}{dx^2} + \cos x = \frac{dy}{dx}$

$\frac{dy}{dx} + \sin x = x^2$

Ordinary Differential Equations:- A Differential Equation containing a single ~~variable~~ independent variable is called the ordinary differential equation.

In the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0$, x is the only one independent variable.

Order of a Differential Equation:- It is the order of the highest ordered derivative occuring in it.

Eg:- $\frac{dy}{dx} + y = 0 \rightarrow$ 1st order.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = 0 \rightarrow$ 2nd order.

$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = 0 \rightarrow$ 3rd order.

Degree of a D.E :- The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

Eg :- $6\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0 \longrightarrow$ Degree 1.

$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + 2y = \sin x \longrightarrow 2^{nd}$ degree.

$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 3y = e^x \longrightarrow 1^{st}$ degree.

Note :- Linear differential equations :- The degree of the differential equation is 1, then it is called a linear differential equation. Linear ODE's of second order : $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + q(x)y = r(x)$

Homogeneous Linear ODE's of 2^{nd} Order :- If $r(x) = 0$ then it is homogeneous.

It can be written in the form,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + q(x)y = 0 \quad \left[\text{or } y'' + P(x)y' + q(x)y = 0\right].$$

Homogeneous 2^{nd} order Linear ODE's with Constant Coefficients :-

It is of the form,

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0} \quad \text{or} \quad \boxed{y'' + ay' + by = 0.}$$

Superposition Principle of homogeneous Linear ordinary Differential Equations :-

If y_1 and y_2 are two solutions of a second order homogeneous linear ODE, then their linear combination $y = c_1y_1 + c_2y_2$ will also be a solution, where c_1 and c_2 are constants.

Eg :- Consider the differential equation

$$y'' + y = 0.$$

Here $y_1 = \cos x$ is a solution of $y'' + y = 0$, since

$$y_1' = -\sin x \qquad \Rightarrow y_1'' + y_1 = -\cos x + \cos x = 0.$$
$$y_1'' = -\cos x$$

Similarly $y_2 = \sin x$ is a solution, since

$$y_2' = \cos x \qquad \Rightarrow y_2'' + y_2 = -\sin x + \sin x = 0.$$
$$y_2'' = -\sin x$$

Then $y = 2\cos x + 3\sin x$ is also a solution.

ie, $y' = 2 \cdot -\sin x + 3\cos x = -2\sin x + 3\cos x$

$y'' = -2\cos x + 3 \cdot -\sin x = -2\cos x - 3\sin x.$

$y'' + y = -2\cos x - 3\sin x + 2\cos x + 3\sin x = 0.$

Linearly dependent and Independent Solutions:-

Two solutions y_1 and y_2 are said to be linearly dependent if $y_1 = k y_2$ or $\frac{y_1}{y_2} = k$, a constant. Otherwise, they are linearly independent.

Eg:- Consider $y'' + y = 0$.

Here $y_1 = \cos x$ and $y_2 = \sin x$ are two solutions.

Here $\frac{y_1}{y_2} = \frac{\cos x}{\sin x} = \cot x$, is not a constant

$\therefore$ y_1 and y_2 are linearly independent.

Que:- Verify by substitution that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the equation $y'' - y = 0$. and are linearly independent.

Ans:- Given $y'' - y = 0$.

$y_1 = e^x \Rightarrow y_1' = e^x$

$y_1'' = e^x.$

$\therefore y_1'' - y = e^x - e^x = 0.$ $\therefore$ y_1 is a solution

$y_2 = e^{-x} \Rightarrow y_2' = -e^{-x}$

$y_2'' = e^{-x}$

$\therefore y_2'' - y_2 = e^{-x} - e^{-x} = 0$

$\therefore y_2$ is also a solution of $y'' - y = 0$

Now, $\frac{y_1}{y_2} = \frac{e^{x}}{e^{-x}} = e^{x} \times e^{+x} = e^{2x} \neq 0$

$\therefore y_1$ and y_2 are linearly independent

Que:- Verify the superposition of the solution of the following ~~solutions of the given~~ differential equations and that they are linearly independent or not.

I) $y_1 = \cos 3x$ and $y_2 = \sin 3x$ are the solutions of the DE.

$y'' + 9y = 0$

Ans:- $y_1 = \cos 3x$ is a solution since

$y_1' = -\sin 3x \times 3$

$y_1'' = -\cos 3x \times 9$

$y'' + 9y =$
$\Rightarrow -9\cos 3x + 9\cos 3x = 0$

$\therefore y_1$ is a solution

Let $y_2 = \sin 3x$

$\Rightarrow y_2' = \cos 3x \times 3$

$y_2'' = -\sin 3x \times 9$

$\therefore y_2'' + 9y = -9\sin 3x + 9\sin 3x = 0$

$\therefore y_2$ is a solution

Now consider $y = ay_1 + by_2 \Rightarrow y = a\cos 3x + b\sin 3x$

Here $y_1' = -3a\sin 3x + 3b\cos 3x$

$y_1'' = -9a\cos 3x - 9b\sin 3x$

$y_1'' + 9y = -9a\cos 3x - 9b\sin 3x + 9a\cos 3x + 9b\sin 3x = 0$

$\therefore y = ay_1 + by_2$ is also a solution

Now, $\frac{y_1}{y_2} = \frac{\cos 3x}{\sin 3x} = \cot 3x \neq$ Constant.

$\therefore$ They are linearly independent

2) $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ be a solution of $y'' + 2y' + y = 0$

→ Initial Value Problem (IVP)

For a second order homogeneous linear ODE with constant coefficient $y'' + ay' + by = 0$, IVP consist of two initial conditions $y(x_0) = k_0$ and $y'(x_0) = k_1$. These conditions are used to determine the arbitrary constants C_1 and C_2 in the general solution $y = C_1 y_1 + C_2 y_2$. The solution thus obtained is called the Particular Solution.

General Solution :- Homogeneous Linear ODE of Second Order :-

Second order Homogeneous linear ODE, consider the equation $y'' + ay' + by = 0$ or

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \longrightarrow (1)$$

$$D^2y + aDy + by = 0$$

$$\rightarrow (D^2 + aD + b)y = 0.$$

To solve this equation, we have to follow the following steps :-

Step 1 :- Write the characteristic equation (auxiliary eq) $\lambda^2 + a\lambda + b = 0 \longrightarrow (2)$, by putting $y'' = \lambda^2$, $y' = \lambda$ and y.

Then solve this quadratic equation (2).

Step 2 :- Let λ_1 and λ_2 be the roots of the characteristic equation called characteristic roots.

Case 1 :- If λ_1 and λ_2 are real and distinct then the solution,

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

Case 2 :- If λ_1 and λ_2 are real and same, say $\lambda_1 = \lambda_2 = \lambda$, then the solution

$$y(x) = (C_1 + C_2 x)e^{\lambda x}.$$

Case 3:- If λ_1 and λ_2 are complex roots, ~~then~~ say $\lambda = \alpha \pm i\beta$, then the solution,

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

Que 1:- Solve the ODE $y'' + 5y' + 6y = 0$

Ans:- Step 1:- Characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0.$$

by solving, we get

$$\lambda = -3, -2.$$

Step 2:- Here $\lambda_1 = -3$ and $\lambda_2 = -2$ are real roots and different.

∴ The solution,

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$\Rightarrow y = c_1 e^{-3x} + c_2 e^{-2x}$$

Que 2:- Solve $y'' - 3y' - 4y = 0.$

Ch. eqn:- $\lambda^2 - 3\lambda - 4 = 0.$

$$\Rightarrow \lambda = -1, 4.$$

the roots are real and different

$$\therefore y = c_1 e^{-x} + c_2 e^{4x}.$$

Que 3:- Solve $y'' + 2y' + y = 0.$

Ans:- The ch. eqn:-

~~$\lambda^2 + 2\lambda + y =$~~

$$\lambda^2 + 2\lambda + 1 = 0.$$

$$\Rightarrow \lambda = -1, -1.$$

The roots are real and same.

$$\therefore y = (c_1 + c_2 x)e^{\lambda x}$$

$$\Rightarrow y = (c_1 + c_2 x)e^{-x}$$

Que 4:- Solve $y'' + 6y' + 9y = 0$

Ans:- Ch. eqn:- $\lambda^2 + 6\lambda + 9 = 0$

$$\Rightarrow \lambda = -3, -3.$$

the roots are same and real.

$$\therefore y = (c_1 + c_2 x)e^{\lambda x}$$

$$\Rightarrow y = (c_1 + c_2 x)e^{-3x}.$$

Que 5:- Solve $y'' + 4y' + 29y = 0.$

Ans:- Ch. eqn $\lambda^2 + 4\lambda + 29 = 0.$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{4^2 - 4\times 1\times 29}}{2\times 1}$$

$$= \frac{-4 \pm \sqrt{16 - 116}}{2}$$

$$= \frac{-4 \pm \sqrt{-100}}{2}$$

$$= \frac{-4 \pm 10i}{2} = 2 \pm 5i.$$

$$\Rightarrow \lambda = \alpha \pm i\beta = 2 \pm 5i.$$

$\Rightarrow \alpha = 2$ and $\beta = 5$.

$\therefore$ The solution

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

$$\Rightarrow y = e^{2x}(C_1 \cos 5x + C_2 \sin 5x)$$

Solution of initial Valued Problems :-

Que 6:- Solve $y'' - y = 0$ given that $y(0) = 2$ and $y'(0) = -2$.

Ans:- Here we have to find the solution of

$$y'' - y = 0.$$

ch. eqⁿ $\lambda^2 - 1 = 0$.

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm 1$

$\therefore$ The roots are

$\lambda_1 = 1$, $\lambda_2 = -1$

The roots are real and different

$\Rightarrow y = C_1 e^{x} + C_2 e^{-x}$.

Given $y(0) = 2$.

$\Rightarrow y(0) = C_1 e^{0} + C_2 e^{0}$

$\Rightarrow y(0) = C_1 + C_2$.

$\Rightarrow C_1 + C_2 = 2 \longrightarrow (1)$.

Also $y'(0) = -2$.

$y' = C_1 e^{x} - C_2 e^{-x}$

$y'(0) = C_1 e^{0} - C_2 e^{0}$

$\Rightarrow y'(0) = C_1 - C_2$.

$\Rightarrow C_1 - C_2 = -2 \longrightarrow (2)$.

Solving (1) and (2)

$$\begin{array}{r} C_1 + C_2 = 2 \quad + \\ C_1 - C_2 = -2 \\ \hline 2C_1 = 0 \end{array}$$

$\Rightarrow C_1 = 0$.

$C_1 + C_2 = 2$

$\Rightarrow C_2 = 2$.

$\therefore y(x) = 0 + 2e^{-x}$

$\therefore y(x) = 2e^{-x}$

Que 7:- Solve $y'' - 3y' + 2y = 0$ given $y(0) = 0$ and $y'(0) = 2$.

Ans:- Given $y'' - 3y' + 2y = 0$.

ch. eqⁿ :- $\lambda^2 - 3\lambda + 2 = 0$.

$\therefore$ the solution $\lambda = 1, 2$.

$\therefore \lambda_1 = 1$ and $\lambda_2 = 2$.

the roots are real and different. The solution,

$\therefore y = C_1 e^{x} + C_2 e^{2x}$.

Now, given $y(0) = 0$

$\Rightarrow y(0) = C_1 e^{0} + C_2 e^{0}$

$\Rightarrow y(0) = C_1 + C_2$

$\Rightarrow C_1 + C_2 = 0 \longrightarrow (1)$

Also $y'(0)=2$.

$y'(x) = c_1e^x + c_2e^{2x}\times 2$.

$\Rightarrow y'(x) = c_1e^x + 2c_2e^{2x}$.

$y'(0) = c_1e^0 + 2c_2e^0$

$\Rightarrow y'(0) = c_1 + 2c_2$

$\therefore c_1 + 2c_2 = 2 \longrightarrow (2)$.

Solving (1) and (2)

$c_1 + c_2 = 0$

$c_1 + 2c_2 = 2$

$0 - c_2 = -2$

$\Rightarrow c_2 = 2$.

$c_1 + c_2 = 0$

$c_1 + 2 = 0$

$\Rightarrow c_1 = -2$.

$\therefore y(x) = -2e^x + 2e^{2x}$

Que 8:- Solve $y'' + 4y' + 5y = 0$

given $y(0) = 2$ $y'(0) = 4$

Ans:- Given $y'' + 4y' + 5y = 0$

ch. eq^n:- $\lambda^2 + 4\lambda + 5 = 0$

Roots are $\lambda_1 = -2 \pm i$

The roots are complex roots.

Here $\alpha = -2$ and $\beta = 1$

$\therefore y = e^{-2x}[c_1\cos x + c_2\sin x]$

Given $y(0) = 2$

$\Rightarrow y(0) = e^0[c_1\cos 0 + c_2\sin 0]$

$\Rightarrow y(0) = 1[c_1\times 1 + 0]$

$\Rightarrow y(0) = c_1$ $\therefore c_1 = 2$.

$y'(x) = e^{-2x}[-c_1\sin x + c_2\cos x]$

$\quad - 2e^{-2x}[c_1\cos x + c_2\sin x]$

$\Rightarrow y'(x) = e^{-2x}[-c_1\sin x + c_2\cos x$

$\quad - 2c_1\cos x - 2c_2\sin x]$

$y'(0) = e^0[-c_1\times 0 + c_2\times 1$

$\quad - 2\times c_1\times 1 - 2c_2\times 0]$

$\Rightarrow y'(0) = e^0[0 + c_2 - 2c_1 - 0]$

$\Rightarrow y'(0) = 1[c_2 - 2c_1]$

$y'(0) = c_2 - 2c_1$

$\Rightarrow c_2 - 2c_1 = 4$

$c_2 - 2\times 2 = 4$

$\Rightarrow c_2 = 8$

$\therefore y(x) = e^{-2x}[2\cos x + 8\sin x]$

Que 9:- Solve $y'' - 4y' + 4y = 0$

given $y(0) = 4$ $y'(0) = 8$

Ans:- Given $y'' - 4y' + 4y = 0$.

ch. eq^n:- $\lambda^2 - 4\lambda + 4 = 0$.

$\Rightarrow \lambda = +2, 2$.

$\therefore$ The roots are Real and same.

$\therefore$ The solution

$y(x) = (c_1 + c_2x)e^{2x}$.

$y(0) = (c_1 + c_2 \times 0) e^0$

$\Rightarrow y(0) = c_1 \Rightarrow c_1 = 4.$

$y'(x) = (c_1 + c_2 x) \times e^{2x} \times 2 + (0 + c_2) . e^{2x}$

$y'(x) = (c_1 + c_2 x) 2e^{2x} + c_2 e^{2x}$

$\Rightarrow y(0) = (c_1 + 0) \times 2e^0 + c_2 e^0$

$\Rightarrow y(0) = 2c_1 + c_2.$

$2c_1 + c_2 = 8.$

$\Rightarrow 2 \times 4 + c_2 = 8$

$\Rightarrow c_2 = 0$

$\therefore y(x) = (4 + 0) e^{2x}$

$\Rightarrow y(x) = 4e^{2x}$

Que 10:- Solve $y'' + 4y = 0$ given $y(0) = 4$ and $y'(0) = 0$.

Ans:- Given $y'' + 4y = 0$.

Charact eq^n:- $\lambda^2 + 4 = 0$

$\Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i.$

The solution is complex numbers. $\alpha = 0, \beta = 2.$

$\therefore y = e^{0x} [c_1 \cos 2x + c_2 \sin 2x]$

$\Rightarrow y(x) = c_1 \cos 2x + c_2 \sin 2x.$

$y(0) = c_1 \times \cos 0 + c_2 \sin 0$

$\Rightarrow y(0) = c_1 \Rightarrow c_1 = 4.$

$y'(x) = c_1 \bar{} \sin 2x \times 2 + c_2 \cos 2x \times 2$

$\Rightarrow y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$

$y'(0) = -2c_1 \times 0 + 2c_2 \times 1$

$0 = \cancel{0} + 2c_2.$

$\Rightarrow c_2 = 0.$

$\therefore y(x) = 4 \cos 2x$

Non-Homogeneous linear ODE with Constant Coefficients :-

Consider the 2nd order non-homogeneous linear ODE with constant coefficient

$$y'' + ay' + by = r(x) \longrightarrow (1)$$

where $r(x)$ is a function in x or constants.

The solution of (1) is

$y = y_h + y_p$, where

y_h is the homogeneous solution and y_p is the particular solution of (1).

There are two methods to find the particular sol^n y_p:-

(i) Method of undetermined Coefficients

(ii) Method of variation of parameters.

1. The Method of Undetermined Coefficients:-

Consider the non-homogeneous ODE is given by:-

$$y'' + ay' + by = r(x), \text{ where } r(x) = ke^{\lambda x}, kx^n,$$

$k\cos\omega x$, $k\sin\omega x$, $ke^{ax}\cos\omega x$, $ke^{ax}\sin\omega x$.

To find the particular solution, we choose a trial solution containing unknown constants which are determined by substitution in the given equation from the following table:-

$r(x)$	Trial solution
1. ke^{ax}	Ae^{ax}
2. kx^n	$A + Bx + Cx^2 + \cdots + Nx^n$.
3. kx^2.	$A + Bx + Cx^2$.
4. $k\cos\omega x$	$A\cos\omega x + B\sin\omega x$
5. $k\sin\omega x$	$A\cos\omega x + B\sin\omega x$
6. $ke^{ax}\cos\omega x$	$e^{ax}(A\cos\omega x + B\sin\omega x)$
7. $ke^{ax}\sin\omega x$	$e^{ax}(A\cos\omega x + B\sin\omega x)$

Que 1:- Solve $y'' + 5y' + 6y = 2e^{-x}$.

Ans:- Step 1:- Find the homogeneous solution from the homogeneous equation $y'' + 5y' + 6y = 0$.

Characteristic eqn:- $\lambda^2 + 5\lambda + 6 = 0$.

$$\Rightarrow \lambda = -3, -2.$$

$\therefore$ The roots are real and different

$$\therefore y_h(x) = c_1e^{-3x} + c_2e^{-2x}$$

Step 2:- Find the non-homogeneous solution y_p from the trial solution.

Here $r(x) = 2e^{-x}$.

$\therefore y_p = Ae^{-x}$

Substitute the particular solution y_p in the eq^n:-

$y'' + 5y' + 6y = 2e^{-x} \Rightarrow y_p'' + 5y_p' + 6y_p = 2e^{-x}$

$y_p' = Ae^{-x} \times -1$ $\qquad$ $y_p'' = -A \cdot e^{-x} \times -1$

$\Rightarrow y_p' = -Ae^{-x}$ $\qquad$ $y_p'' = Ae^{-x}$.

$\therefore Ae^{-x} + 5 \times -Ae^{-x} + 6 \times Ae^{-x} = 2e^{-x}$

$\Rightarrow e^{-x}(A - 5A + 6A) = 2e^{-x}$

$\Rightarrow 2Ae^{-x} = 2e^{-x} \Rightarrow 2A = 2 \Rightarrow \underline{A = 1}$.

$\therefore y_p = 1 \times e^{-x} \Rightarrow y_p = e^{-x}$.

$\therefore$ The general sol^n:- $y = y_p + y_h$

$\Rightarrow y = c_1e^{-3x} + c_2e^{-2x} + e^{-x}$

Que2:- Solve $y'' - y = 3\cos x$

Ans:- Step 1:- Homogeneous solution y_h:-

Consider $y'' - y = 0$.

Characteristic equation $\lambda^2 - 1 = 0$.

$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\therefore$ roots are real and different.

$\therefore y_h(x) = c_1e^{x} + c_2e^{-x}$

Step 2:- Particular solution y_p:-

Here $r(x) = 3\cos x$.

$\Rightarrow y_p = A\cos x + B\sin x$

Substitute y_p in the eq^n: $y'' - y = 3\cos x$.

$y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$.

$y_p'' = -A\cos x - B\sin x$.

$\Rightarrow (-A\cos x - B\sin x) - (A\cos x + B\sin x) = 3\cos x$

$\Rightarrow -A\cos x - B\sin x - A\cos x - B\sin x = 3\cos x$.

$\Rightarrow -2A\cos x - 2B\sin x = 3\cos x$.

Comparing the coefficients :- $-2A = 3 \Rightarrow A = -3/2$

$-2B = 0 \Rightarrow B = 0$.

$\therefore y_p = \frac{-3}{2}\cos x$

$\therefore$ The general sol^n:- $y = C_1 e^{x} + C_2 e^{-x} - \frac{3}{2}\cos x$

Que3:- solve $y'' + 3y' + 2y = 12x^2$.

Ans:- Step 1:- find y_h :- H.E $y'' + 3y' + 2y = 0$.

ch. eq^n:- $\lambda^2 + 3\lambda + 2 = 0$.

$\therefore$ we get $\lambda = -2, -1$.

The roots are real and different

$\Rightarrow y_h = C_1 e^{-2x} + C_2 e^{-x}$

Step 2:- find y_p:-

Here $r(x) = 12x^2$.

$\Rightarrow y_p = A + Bx + Cx^2$.

$y_p' = 0 + B + 2Cx = B + 2Cx$

$y_p'' = 0 + 2C = 2C$.

Sub in $y'' + 3y' + 2y = 12x^2$

$\Rightarrow 2C + 3\times(B + 2Cx) + 2(A + Bx + Cx^2) = 12x^2$

$\Rightarrow 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 = 12x^2$

$\Rightarrow 2Cx^2 + (6C + 2B)x + (2C + 2A + 3B) = 12x^2$.

Compairing the equations; we get

$2C = 12$
$\Rightarrow C = 12/2 = 6$

$6C + 2B = 0$
$6 \times 6 + 2B = 0$
$\Rightarrow 36 + 2B = 0$
$2B = -36$
$B = -18$

$2C + 3B + 2A = 0$
$(2\times 6) + (3\times -18) + 2A = 0$
$12 - 54 + 2A = 0$
$2A = 42$
$\Rightarrow A = 21$

$\therefore y_p = 21 + -18x + 6x^2$

$\therefore y = C_1 e^{-2x} + C_2 e^{-x} + 21 - 18x + 6x^2$

Que 4:- Solve $y'' + 4y' + 4y = e^{-x}\cos x$.

Ans:- Step 1:- find y_h:

H.E :- $y'' + 4y' + 4y = 0$, A.E $\lambda^2 + 4\lambda + 4 = 0$
$\Rightarrow \lambda = -2, -2$

the roots are real and same.

$\therefore$ The homogeneous solution $y_h = (C_1 + C_2 x)e^{-2x}$.

Step 2:- Non-homogeneous soln:- y_p

$r(x) = e^{ax}.\cos bx$.
$y_p = e^{-x}[A\cos x + B\sin x]$

Let $y_p = e^{-x}[A\cos x + B\sin x]$

$y_p' = e^{-x} \times -1 [A\cos x + B\sin x] + e^{-x}[-A\sin x + B\cos x]$

$\Rightarrow y_p' = -e^{-x}[A\cos x + B\sin x] + e^{-x}[-A\sin x + B\cos x]$

$\Rightarrow y_p' = e^{-x}[-A\cos x - B\sin x - A\sin x + B\cos x]$

$y_p'' = e^{-x} \times -1 [-A\cos x - B\sin x - A\sin x + B\cos x] + e^{-x}[A\sin x - B\cos x - A\cos x - B\sin x]$

$y_p'' = e^{-x}[\cancel{A\cos x} + \cancel{B\sin x} + A\sin x - B\cos x + A\sin x - B\cos x - \cancel{A\cos x} - \cancel{B\sin x}]$

$$\Rightarrow y_p'' = e^{-x}[2A\sin x - 2B\cos x]$$

$$\therefore y_p'' + 4y_p' + 4y_p = e^{-x}\cos x$$

$$\Rightarrow e^{-x}[2A\sin x - 2B\cos x] + 4e^{-x}[-A\cos x - B\sin x - A\sin x + B\cos x] + 4e^{-x}[A\cos x + B\sin x] = e^{-x}\cos x$$

$$\Rightarrow 2Ae^{-x}\sin x - 2Be^{-x}\cos x - 4Ae^{-x}\cos x - 4Be^{-x}\sin x - 4Ae^{-x}\sin x + 4Be^{-x}\cos x + 4Ae^{-x}\cos x + 4Be^{-x}\sin x = e^{-x}\cos x$$

$$e^{-x}\sin x[2A - 4B - 4A + 4B] + e^{-x}\cos x[-2B - 4A + 4B + 4A] = e^{-x}\sin x$$

$$\Rightarrow e^{-x}\sin x \times (-2A) + e^{-x}\cos x(2B) = e^{-x}\cos x$$

Comparing, $-2A = 0$, $2B = 1$

$$\Rightarrow A = 0 \text{ and } B = 1/2.$$

$$\therefore y_p = e^{-x}\left[0 + \frac{1}{2}\sin x\right]$$

$$\therefore y_p = \frac{1}{2}e^{-x}\sin x.$$

$\therefore$ General solution $y = (C_1 + C_2 x)e^{-2x} + \frac{1}{2}e^{-x}\sin x$

Modification Rule:- If any term in the trial solution appears in the Complementary solution, we multiply this trial solution by x. If this solution corresponds to a double root of characteristic eqn, multiply by x^2 and so on.

Que 5:- Solve $(D^2+1)y = \sin x$.

Ans:- Given $(D^2+1)y = \sin x$

$$\Rightarrow D^2y + y = \sin x$$

$$\Rightarrow y'' + y = \sin x$$

Step 1:- y_h?

H.E :- $y'' + y = 0$.

C.h.E $\Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$

Here $a = 0$ and $b = 1$

$\therefore$ Homogeneous Solution :-

$$y_h = e^{0x}\left[c_1 \cos x + c_2 \sin x\right]$$

$$\therefore y_h = c_1 \cos x + c_2 \sin x$$

Step 2:- y_p?

Here $r(x) = \sin x$ be a function in y_h.

$$\therefore y_p = x(A\cos x + B\sin x)$$

$$y_p' = 1(A\cos x + B\sin x) + x(-A\sin x + B\cos x)$$

$$y_p' = A\cos x + B\sin x + x(-A\sin x + B\cos x)$$

$$y_p'' = -A\sin x + B\cos x + 1(-A\sin x + B\cos x) + x(-A\cos x + B\sin x)$$

$$\Rightarrow y_p'' = -A\sin x + B\cos x - A\sin x + B\cos x - x(A\cos x + B\sin x)$$

$$\Rightarrow y_p'' = -2A\sin x + 2B\cos x - x(A\cos x + B\sin x)$$

$$\therefore y_p'' + y_p = \sin x$$

$$\Rightarrow -2A\sin x + 2B\cos x - x(A\cos x + B\sin x) + x(A\cos x + B\sin x) = \sin x$$

$$\Rightarrow -2A\sin x + 2B\cos x = \sin x$$

Compairing $\Rightarrow -2A = 1$, $2B = 0$

$\Rightarrow A = -1/2$ and $B = 0$

$$\therefore y_p = x(-1/2 \cos x + 0) = -1/2\, x\cos x.$$

$$\therefore y = c_1 \cos x + c_2 \sin x - 1/2\, x\cos x$$

Que6:- Solve $y'' + 2y' + y = e^{-x}$, given $y(0) = 1$, $y'(0) = 2$

Ans:- Step 1:- y_h?

H.E :- $y'' + 2y' + y = 0$.

A.E :- $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1, -1$

$\therefore$ The roots are real and same.

i.e, the Homogeneous sol^n $y_h = (c_1 + c_2 x)e^{-x}$.

Step 2:- y_p?

Here $r(x) = e^{-x}$ which is a double root of y_h

$\Rightarrow y_p = x^2(Ae^{-x})$

$y_p' = 2x(Ae^{-x}) + x^2 \times A \times e^{-x} \times -1$

$y_p' = 2x(Ae^{-x}) - x^2(Ae^{-x})$

$y_p'' = 2(Ae^{-x}) + 2x(Ae^{-x} x -1) - (2x(Ae^{-x}) + x^2(Ae^{-x} x -1))$

$\Rightarrow y_p'' = 2Ae^{-x} - 2xAe^{-x} - 2xAe^{-x} + x^2 Ae^{-x}$

$y_p'' = 2Ae^{-x} - 4Axe^{-x} + Ax^2 e^{-x}$

$\therefore y_p'' + 2y_p' + y = e^{-x}$

$\Rightarrow 2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x} + 2(2Axe^{-x} - x^2Ae^{-x})$
$+ x^2(Ae^{-x}) = e^{-x}$

$\Rightarrow 2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x} + 4Axe^{-x} - 2x^2Ae^{-x}$
$+ Ax^2e^{-x} = e^{-x}$

$\Rightarrow 2Ae^{-x} = e^{-x} \Rightarrow 2A = 1 \Rightarrow A = 1/2$.

$\therefore y_p = x^2(\tfrac{1}{2}e^{-x}) \Rightarrow y_p = \tfrac{1}{2}x^2e^{-x}$.

$\therefore y = (c_1 + c_2 x)e^{-x} + \tfrac{1}{2}x^2 e^{-x}$

Here $y(x) = (c_1 + c_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$.

$y(0) = 1 \Rightarrow y(0) = (c_1 + 0)e^0 + \frac{1}{2} \times 0$

$\Rightarrow y(0) = c_1 = 1$.

$\therefore y(x) = (1 + c_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$.

$y'(x) = (0 + c_2)e^{-x} + (1 + c_2 x)e^{-x} \times -1 + \frac{1}{2} \times 2x \cdot e^{-x} + \frac{1}{2} x^2 e^{-x} \times -1$

$\Rightarrow y'(0) = c_2 + (1+0)e^0 \times -1 + \frac{1}{2} \times 0 + \frac{1}{2} \times 0$

$\Rightarrow y'(0) = c_2 + 1 \times -1 + 0 \Rightarrow y'(0) = c_2 - 1 = 2$.

~~$\Rightarrow y'(0) =$~~ $c_2 = 3$.

Que 7:- Solve $y'' - 4y' + 3y = e^x - \frac{9}{2}x$, given $y(0) = 0$, $y'(0) = 6$

Ans:- **Step 1:-** y_h?

H.E:- $y'' - 4y' + 3y = 0$.

Ch. eq^n:- $\lambda^2 - 4\lambda + 3 = 0$.

$\Rightarrow \lambda = 3, 1$

The roots are real and different.

$\rightarrow y_h = c_1 e^x + c_2 e^{3x}$

Step 2:- y_p?

Here $r(x) = e^x - \frac{9}{2}x$. e^x is a function in y_h, then

$y_p = Axe^x + (Bx + C)$

$y_p' = Ae^x + Axe^x + B$

$y_p'' = Ae^x + Ae^x + Axe^x + 0$

$\Rightarrow y_p'' = 2Ae^x + Axe^x$.

$\therefore y'' - 4y' + 3y = e^x - \frac{9}{2}x$

$$2Ae^x + Axe^x - 4(Ae^x + Axe^x + B) + 3(Axe^x + Bx + C) = e^x - \tfrac{9}{2}x$$

$$\Rightarrow 2Ae^x + Axe^x - 4Ae^x - 4Axe^x - 4B + 3Axe^x + 3Bx + 3C = e^x - \tfrac{9}{2}x$$

$$\Rightarrow e^x(2A - 4A) - 4B + 3C + 3Bx = e^x - \tfrac{9}{2}x$$

Comparing, $2A - 4A = 1$ $\quad$ $3B = -\tfrac{9}{2}$ $\quad$ $-4B + 3C = 0$

$\Rightarrow -2A = 1$ $\quad$ $\Rightarrow B = -\tfrac{9}{2\times 3}$ $\quad$ $-4 \times \tfrac{-3}{2} + 3C = 0$

$\Rightarrow A = -\tfrac{1}{2}$ $\quad$ $\Rightarrow B = -\tfrac{3}{2}$ $\quad$ $6 + 3C = 0$

$3C = -6$

$\Rightarrow C = -2$

$$y_p = -\tfrac{1}{2}xe^x + -\tfrac{3}{2}x + (-2)$$

$$\therefore y = c_1e^x + c_2e^{3x} - \tfrac{1}{2}xe^x - \tfrac{3}{2}x - 2$$

$$y(0) = c_1 + c_2 - 0 - 0 - 2 \Rightarrow c_1 + c_2 - 2 = 0$$
$$\Rightarrow c_1 + c_2 = 2 \longrightarrow (1)$$

$$y'(x) = c_1e^x + 3c_2e^{3x} - \left[\tfrac{1}{2}e^x + \tfrac{1}{2}xe^x\right] - \tfrac{3}{2} - 2$$

$$y'(0) = c_1 + 3c_2 - \left[\tfrac{1}{2} + 0\right] - \tfrac{3}{2} - 2$$
$$= c_1 + 3c_2 - \tfrac{1}{2} - \tfrac{3}{2} - 2$$
$$= c_1 + 3c_2 - 2 - 2 \Rightarrow c_1 + 3c_2 - 4 = 6$$
$$\Rightarrow c_1 + 3c_2 = 10 \longrightarrow (2)$$

Solving (1) and (2)

$$\begin{array}{r} c_1 + 3c_2 = 10 \\ - \quad c_1 + c_2 = 2 \\ \hline 2c_2 = 8 \end{array}$$
$$\Rightarrow c_2 = 4$$

$c_1 + c_2 = 2$

$c_1 + 4 = 2$

$\Rightarrow c_1 = -2$

$\therefore$ The general solution,

$$y = -2e^x + 4e^{3x} - \tfrac{1}{2}xe^x - \tfrac{3}{2}x - 2$$

(2) The Method Variation of Parameters

Consider the second order linear ODE with constant coefficients, $y'' + ay' + by = r(x)$. $\longrightarrow$ ①

Let the complementary function or homogeneous solution be, $y_h = c_1 y_1 + c_2 y_2 \longrightarrow$ ②.

Then the particular integral, $y_p = u(x) y_1 + v(x) y_2$,

where $u(x) = -\int \frac{y_2 \cdot r(x)}{W} dx$ and $v(x) = \int \frac{y_1 \, r(x)}{W} . dx$

When $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of y_1 and y_2.

Que 1:- Solve $y'' + y = \operatorname{cosec} x$.

Step 1:- y_h?

H.E : $y'' + y = 0$.

Ch. eq^n :- $\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$

$\Rightarrow \lambda = \pm i$.

$\therefore$ the roots are complex roots., $a = 0, b = 1$.

$\therefore y_h = e^{0x} [c_1 \cos x + c_2 \sin x]$

$\Rightarrow y_h = c_1 \cos x + c_2 \sin x$.

Step 2:- y_p? $y_p = u(x) y_1 + v(x) y_2$.

Here $y_1 = \cos x$, $y_2 = \sin x$ and $r(x) = \operatorname{cosec} x$.

Now, $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$\therefore u(x) = -\int \frac{y_2 \, r(x)}{W} dx = -\int \frac{\sin x \times \operatorname{cosec} x}{1} dx$

$= -\int \sin x \times \frac{1}{\sin x} dx = -\int 1 \, dx = -x$

$$V(x) = \int \frac{y_1\, r(x)}{W}\, dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\, dx$$

$$= \int \cos x \cdot \frac{1}{\sin x}\, dx = \int \frac{\cos x}{\sin x}\, dx$$

$$= \int \cot x\, dx = \log(\sin x).$$

$$\therefore y_p = -x\cos x + \log(\sin x).\sin x$$

$$\Rightarrow y(x) = c_1\cos x + c_2\sin x - x\cos x + \log(\sin x).\sin x$$

Que2:- solve $y'' - 4y' + 5y = \frac{e^{2x}}{\sin x}$

Ans:- Step 1:- y_h?

H.E:- $y'' - 4y' + 5y = 0$.

Ch. eqn:- $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = 2 \pm i$, $a = 2,\ b = 1$

$$\therefore y_h = e^{2x}\left[c_1\cos x + c_2\sin x\right]$$

$$\Rightarrow y_h = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x$$

Step 2:- y_p?

Here $y_1 = e^{2x}\cos x$, $y_2 = e^{2x}\sin x$ and $r(x) = \frac{e^{2x}}{\sin x}$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x}.\cos x & e^{2x}.\sin x \\ \frac{d}{dx}(e^{2x}.\cos x) & \frac{d}{dx}(e^{2x}.\sin x) \end{vmatrix}$

$$= e^{2x}.\cos x.\frac{d}{dx}(e^{2x}.\sin x) - \frac{d}{dx}(e^{2x}.\cos x).\ e^{2x}.\sin x$$

$$= e^{2x}.\cos x\left[2e^{2x}\sin x + e^{2x}.\cos x\right] - \left[2e^{2x}\cos x - e^{2x}.\sin x\right] e^{2x}.\sin x$$

$$= 2e^{4x}\cos x.\sin x + e^{4x}\cos^2 x - 2e^{4x}.\cos x.\sin x + e^{4x}\sin^2 x.$$

$$= e^{4x}\left[\cos^2 x + \sin^2 x\right]$$

$$= e^{4x}$$

Now, $y_p = u(x) y_1 + v(x) y_2$

where $u(x) = -\int \frac{y_2 R(x)}{W} dx = -\int \frac{e^{2x}\sin x \cdot \frac{e^{2x}}{\sin x}}{e^{4x}} dx$

$= -\int \frac{e^{4x}}{e^{4x}} dx = -\int 1\, dx = -x$

and $v(x) = \int \frac{y_1 r(x)}{W} dx = \int \frac{e^{2x}\cos x \cdot \frac{e^{2x}}{\sin x}}{e^{4x}} dx$

$= \int \frac{e^{4x} \cdot \cot x}{e^{4x}} dx = \int \cot x\, dx = \log(\sin x)$

$\therefore y_p = -x e^{2x}\cos x + \log(\sin x) e^{2x} \sin x.$

$\Rightarrow y(x) = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x - x e^{2x}\cos x + e^{2x}\log(\sin x) \cdot \sin x$

Que 3:- Solve $y'' + 4y = \tan 2x.$

Ans:- Step 1:- y_h?

H.E: $y'' + 4y = 0.$

ch. eqn:- $\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i$

$\therefore$ roots are complex numbers, $a = 0$, $b = 2$.

$\therefore y_h = e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$

$\Rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x.$

Step 2:- y_p?

Here $r(x) = \tan 2x$, $y_1 = \cos 2x$, $y_2 = \sin 2x$.

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2.$$

Now, $y_p = u(x) y_1 + v(x) y_2.$

where $u(x) = -\int \frac{y_2\, r(x)}{W}\,dx = -\int \frac{\sin 2x \cdot \tan 2x}{2}\,dx$

$$= \frac{-1}{2}\int \sin 2x \times \frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx.$$

$$= \frac{-1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx \qquad [\sin^2 x + \cos^2 x = 1]$$

$$= -\frac{1}{2}\int \frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int (\sec 2x - \cos 2x)\,dx$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x$$

and $v(x) = \int \frac{y_1\, r(x)}{W}\,dx = \int \frac{\cos 2x \cdot \tan 2x}{2}\,dx$

$$= \frac{1}{2}\int \cos 2x \cdot \frac{\sin 2x}{\cos 2x}\,dx = \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}\,\frac{-\cos 2x}{2} = -\frac{\cos 2x}{4}.$$

$$\therefore y_p = \left[-\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x\right]\cdot \cos 2x$$

$$- \frac{\cos 2x}{4} \times \sin 2x.$$

$$y_p = -\frac{1}{4}\cos 2x \cdot \log(\sec 2x + \tan 2x) + \cancel{\frac{2\cos 2x \cdot \sin 2x}{4}}\ 0$$

$$\therefore y(x) = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\,\log(\sec 2x + \tan 2x)$$

$$\cancel{-\frac{2\cos 2x \cdot \sin 2x}{4}}$$

$\sin x$

Que 4: Solve $y'' + 4y = \sec 2x$.

Ans: Step 1: y_h?

H.E: $y'' + 4y = 0$.

Ch. eqn: $\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i$.

$\therefore y_h = e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$

$\Rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x$.

Step 2: y_p?

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $r(x) = \sec 2x$.

Now, $W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$.

and $y_p = u(x) y_1 + v(x) y_2$

where $u(x) = -\int \frac{y_2 r(x)}{W} dx = -\int \frac{\sin 2x \cdot \sec 2x}{2} dx$

$= \frac{-1}{2}\int \sin 2x \frac{1}{\cos 2x} dx = \frac{-1}{2}\int \tan 2x \, dx$

$= \frac{-1}{2} \log \frac{(\sec 2x)}{2} = \frac{-1}{4} \log(\sec 2x)$.

$v(x) = \int \frac{y_1 r(x)}{W} dx = \int \frac{\cos 2x \cdot \sec 2x}{2} dx$

$= \frac{1}{2}\int \cos 2x \times \frac{1}{\cos 2x} dx = \frac{1}{2}\int 1 dx = \frac{1}{2} \cdot x$

$\therefore y_p = -\frac{1}{4} \log(\sec x) \cdot \cos 2x + \frac{1}{2} x \cdot \sin 2x$.

$\therefore y = c_1 \cos 2x + c_2 \sin 2x - \frac{\cos 2x}{4} \log(\sec 2x) + \frac{x}{2} \sin 2x$

Ques 5:- Solve $y'' + y = x \sin x$

Ans:- Step 1: y_h

H.E: $y'' + y = 0$.

Ch. eqn:- $\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$.

roots are complex roots, $a = 0$, $b = 1$

$$\therefore y_h = e^{0x}[c_1 \cos x + c_2 \sin x]$$

$$\Rightarrow y_h = c_1 \cos x + c_2 \sin x$$

Step 2:- y_p?

Here $r(x) = x \sin x$, $y_1 = \cos x$, $y_2 = \sin x$

Now, $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$.

$y_p = u(x) y_1 + v(x) y_2$,

where $u(x) = -\int \frac{y_2 r(x)}{W} dx$

$$= -\int \frac{\sin x \cdot x \sin x}{1} dx = -\int \frac{x \sin^2 x}{1} dx$$

$$= -\int x \sin^2 x \, dx$$

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos 2x &= 1 - 2\sin^2 x \end{aligned}}$$

$$= -\int x \left[\frac{1 - \cos 2x}{2}\right] dx$$

$$= -\frac{1}{2} \int x - x \cos 2x \, dx$$

$$= -\frac{1}{2}\left[\frac{x^2}{2} - x \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2} dx\right]$$

$$= -\frac{1}{2}\left[\frac{x^2}{2} - \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= -\frac{1}{2}\left[\frac{2x^2 - 2x \sin x + \cos 2x}{4}\right]$$

$$= \frac{-1}{8}\left[2x^2 - 2x \sin x + \cos 2x\right]$$

$$v(x) = \int \frac{y_1\, r(x)}{W}\,dx = \int \frac{\cos x \cdot x \sin x}{1}\,dx$$

$$= \int \frac{x \cdot \cos x \sin x}{1}\,dx = \int \frac{x \cdot 2\cos x \sin x}{2}\,dx$$

$$= \int x \frac{\sin 2x}{2}\,dx = \frac{1}{2}\int x \sin 2x\,dx$$

$$= \frac{1}{2}\left[x \cdot \frac{-\cos 2x}{2} - \int 1 \cdot \frac{-\cos 2x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= \frac{1}{8}\left[-2x\cos 2x + \sin 2x\right]$$

$$\therefore y_p = -\frac{1}{8}\left[2x^2 - 2x\sin 2x + \cos 2x\right] + \frac{1}{8}\left[-2x\cos 2x + \sin 2x\right]$$

$$\therefore y = c_1\cos x + c_2 \sin x - \frac{1}{8}\left(2x^2 - 2x\sin 2x + \cos 2x\right) + \frac{1}{8}\left(-2x\cos 2x + \sin 2x\right)$$