11/2/25

MODULE 2:

power series

An infinite series of a complex no. of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1 (z-z_0)^1 + a_2 (z-z_0)^2 + \cdots + a_n (z-z_0)^n + \cdots$$

where z_0 and the coefficients a_n are complex constants. and z may be any point in a stated region containing z_0 is known as power series.

$\boxed{(z-z_0)^0 = 1}$

Radius of convergence of the series

Radius of convergence is denoted and defined by $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ OR $R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$

⇒ power [illegible]

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

A: here $a_n = \frac{1}{n!}$ [coefficient of z^n]

$\therefore a_{n+1} = \frac{1}{(n+1)!}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{\left(\frac{1}{n!}\right)}{\frac{1}{(n+1)!}} = \frac{1}{n!} \times \frac{(n+1)!}{1}$$

$(n+1)! = n!(n+1)$

$(n+2)! = n!(n+1)(n+2)$

$$= \frac{1}{n!} \times (n+1)!$$

$$= \frac{1}{\cancel{n!}} \times \cancel{n!}(n+1)$$

~~∴ radius of convergence~~

$$= \underline{n+1}$$

$\therefore$ radius of convergence $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$\boxed{\infty + 1} = \infty$

$$= \lim_{n\to\infty} |n+1|$$

$$= \lim_{n\to\infty} (n+1) = \infty + 1$$

$$= \underline{\underline{\infty}}$$

2. Find the radius of convergence $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-$

let $a_n = \frac{(2n)!}{(n!)^2}$ $\boxed{\sum_{n=0}^{\infty} a_n (z-z_0)^n}$

$$a_{n+1} = \frac{2(n+1)!}{[(n+1)!]^2} = \frac{(2n+2)!}{[(n+1)!]^2}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(2n)!}{(n!)^2} \times \frac{[(n+1)!]^2}{(2n+2)!}$$

$$= \frac{\cancel{(2n)!}}{(n!)^2} \times \frac{[n!(n+1)]^2}{\cancel{(2n)!}(2n+1)(2n+2)}$$

$$= \frac{\cancel{(n!)^2}(n+1)^2}{\cancel{(n!)^2}(2n+1)(2n+2)}$$

$\boxed{1 = \frac{1}{n}}$ n common outs

$$= \frac{\left[n\left(1+\frac{1}{n}\right)\right]^2}{n\left(2+\frac{1}{n}\right)n\left(2+\frac{2}{n}\right)}$$

$$= \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{\cancel{n^2}\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}$$

$\therefore$ radius of convergence $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$R = \lim_{n\to\infty} \frac{(1+1/n)^2}{(2+1/n)(2+2/n)}$$

$$= \frac{(1+1/\infty)^2}{(2+1/\infty)(2+2/\infty)}$$

$$= \frac{(1+0)^2}{(2+0)(2+0)} = \frac{1}{2\times 2}$$

$$= \frac{1}{4}$$

$$\left.\begin{array}{l} \frac{1}{\infty} = \\ \frac{2}{\infty} = \\ \frac{3}{\infty} = \end{array}\right\} 0$$

$$\frac{\infty}{\infty} = \infty$$

3. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} (1+1/n)^{n^2} z^n$

A: let $a_n = (1+1/n)^{n^2}$

$\therefore$ Radius of convergence $R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$

$$= \frac{1}{\lim_{n\to\infty} \left[(1+1/n)^{n^{\cancel{2}}}\right]^{1/\cancel{n}}}$$

$$\boxed{\lim_{n\to\infty} (1+1/n)^n = e}$$

$$= \frac{1}{\lim_{n\to\infty} \left[(1+1/n)^n\right]}$$

$$= \frac{1}{e}$$

4. Find the radius of convergence of $\sum_{n=0}^{\infty} n^n z^n$

$a_n = n^n$

$$R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty} (n^{\cancel{n}})^{1/\cancel{n}}}$$

$$= \frac{1}{\lim_{n\to\infty} n}$$

$$= \frac{1}{\infty} = 0$$

1. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$

$$\boxed{\sum_{n=0}^{\infty} a_n (z-z_0)^n}$$

let $a_n = \frac{n!}{n^n}$

$\therefore a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{\cancel{n!}}{n^n} \times \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{n!}\ \cancel{(n+1)}}$$

$$= \frac{(n+1)^n}{n^n} = \frac{[n(1+1/n)]^n}{n^n}$$

$$= \frac{\cancel{n^n}(1+1/n)^n}{\cancel{n^n}}$$

$$\therefore R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} (1+1/n)^n = e$$

$$\underline{\underline{R = e}}$$

2. $\sum_{n=0}^{\infty} \frac{(3+4i)^n}{5^n}$

$a_n = \frac{1}{5^n}$

$$R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

$$= \frac{1}{\lim_{n\to\infty} \left(\frac{1}{5^n}\right)^{1/n}} = \frac{1}{\lim_{n\to\infty} \frac{1^{1/n}}{(5^{n})^{1/n}}}$$

lim of a constant is constant itself

$$= \frac{1}{\lim_{n\to\infty} \left(\frac{1}{5}\right)}$$

$$= \frac{1}{\left(\frac{1}{5}\right)} = 5$$

Taylor's Series

let $f(z)$ be analytic in a domain (D) and let $z = a$ be any point in D ; then $f(z)$ can be expressed as

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a)^1 + a_2(z-a) + \dots \quad n = 0, 1, 2 \dots}$$

Results:

* $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$ (alternate sign)
* $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$ (same ")
* $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
* $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$
* $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ (even nos.)
* $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ (odd nos.)

Q1 Expand $f(z) = \frac{1}{z+1}$ as a taylor series about :

(i) $z = 3$ | (ii) $z = -2$

① ⇒ Taylor series about $z = 3$

ie, here $a = 3$

$\therefore (z-a) = (z-3)$

$f(z) = \sum a_n (z-a)^n$

$\therefore z - a = z - 3$

given that : $f(z) = \frac{1}{z+1}$

$$= \frac{1}{(z-3)+(1+3)}$$ [adding & subtracting by 3]

ie, $f(z) = \frac{1}{(z-3)+4} \rightarrow 1$

$$= \frac{1}{4\left[1 + \frac{z-3}{4}\right]}$$

$$= \frac{1}{4}\left[1 + \underbrace{\left(\frac{z-3}{4}\right)}_{x}\right]^{-1}$$

$\Rightarrow (1+x)^{-1}$ $\qquad x = \frac{z-3}{4}$

$$= \frac{1}{4}\left[1 - \left(\frac{z-3}{4}\right) + \left(\frac{z-3}{4}\right)^2 - \left(\frac{z-3}{4}\right)^3 + \dots\right]$$

② ⇒ Taylor series about $z = -2$

here $a = -2$

$(z - a) = (z - -2)$

$= (z+2)$

$F(z) = \frac{1}{z+1}$

$= \cancel{\frac{1}{(z+2)+(z-2)}} -$

$= \frac{1}{(z+2)-2+1} = \frac{1}{(z+2)-1}$

$$= \frac{1}{-[1-(z+2)]}$$

$$= -[1-(z+2)]^{-1} \qquad x = z+2$$

$$= -[1 + (z+2) + (z+2)^2 + (z+3)^3 + \cdots]$$

$$= -1 - (z+2) - (z+2)^2 - (z+3)^3 + \cdots$$

2. Expand $f(z) = \frac{1}{z^2}$, as a taylor sereis about $z=3$

Taylor series about $z = 3$

$a = 3$

$\therefore (z-a) = (z-3)$

given that $f(z) = \frac{1}{z^2}$

$$= \frac{1}{[(z-3)+3]^2}$$

$$= \frac{1}{\left\{3\left[1+\left(\frac{z-3}{3}\right)\right]\right\}^2}$$

$$= \frac{1}{3^2\left[1+\frac{z-3}{3}\right]^2}$$

$$= \frac{1}{9\left[1+\left(\frac{z-3}{3}\right)\right]^2} = \frac{1}{9}\underbrace{\left[1+\left(\frac{z-3}{3}\right)\right]^{-2}}_{(1+x)^{-2}}, \quad x = \frac{z-3}{3}$$

$$= \frac{1}{9}\left\{1 - 2\left(\frac{z-3}{3}\right) + 3\left(\frac{z-3}{3}\right)^2 - 4\left(\frac{z-3}{3}\right)^3 + \cdots\right\}$$

1\. Expand $f(z) = \cos z$ in taylor series $z = \pi/4$

A: Here $a = \pi/4$

$\therefore z-a = z - \pi/4$

$$f(z) = \cos z$$

$$= \cos\left[\underbrace{(z-\pi/4)}_{A} + \underbrace{\pi/4}_{B}\right]$$

(adding & subtracting by $\pi/4$)

$$\boxed{\cos A+B = \cos A \cos B - \sin A \sin B}$$

$$f(z) = \cos(z-\pi/4) \cdot \cos \pi/4 - \sin(z-\pi/4) \cdot \sin \pi/4$$

$$\boxed{\cos \pi/4 = \frac{1}{\sqrt{2}} \quad \sin \pi/4 = \frac{1}{\sqrt{2}}}$$

$$= \cos(z-\pi/4) \cdot \frac{1}{\sqrt{2}} - \sin(z-\pi/4) \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\cos\left(z-\frac{\pi}{4}\right) - \sin\left(z-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 - \frac{\left(z-\frac{\pi}{4}\right)^2}{2!} + \frac{\left(z-\frac{\pi}{4}\right)^4}{4!} - \cdots\right.$$

$$\left. - \frac{\left(z-\frac{\pi}{4}\right)}{1!} - \frac{\left(z-\frac{\pi}{4}\right)^3}{3!} + \frac{\left(z-\frac{\pi}{4}\right)^5}{5!} \cdots\right]$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

$$\therefore f(z) = \frac{1}{\sqrt{2}}\left[1 - \frac{(z-\pi/4)^2}{2!} + \frac{(z-\pi/4)^4}{4!} - \cdots - \frac{(z-\pi/4)}{1!} + \frac{(z-\pi/4)^3}{3!} - \frac{(z-\pi/4)^5}{5!} + \cdots\right]$$

2. Expand $f(z) = \sin z$ in taylor series $z = \pi/4$

Here, $a = \pi/4$

$$z - a = z - \pi/4$$

$$f(z) = \sin z$$

$$= \sin\left[\underbrace{(z - \pi/4)}_{A} + \underbrace{\pi/4}_{B}\right]$$

$$f(z) = \sin(z - \pi/4)\cos\pi/4 + \cos(z-\pi/4)\sin\pi/4$$

$$= \sin\left(z - \frac{\pi}{4}\right)\cdot \cancel{\cos}\frac{1}{\sqrt{2}} + \cos\left(z - \frac{\pi}{4}\right)\cancel{\sin}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\sin\left(z - \frac{\pi}{4}\right) + \cos\left(z - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{\left(z - \frac{\pi}{4}\right)}{1!} - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^5}{5!} \cdots + 1 - \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} + \frac{\left(z - \frac{\pi}{4}\right)^4}{4} - \cdots\right]$$

Note: Taylor series of a function $f(z)$ is given by, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

where $a_n = \frac{1}{n!} f^n(a)$

$$\therefore a_0 = \frac{1}{0!} f^0(a) = 1$$

$$a_1 = \frac{1}{1!} f'(a) \leftarrow \text{1st derivative}$$

$$a_2 = \frac{1}{2!} f^2(a) \text{ — 2nd derivative}$$

$$a_3 = \frac{1}{3!} f^3(a)$$

- - - - - - - - -

Maclaurin's Series

The maclaurin's series expansion of $f(z)$ is obtained by putting $a=0$ in taylor series and is given by,

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \ldots$$

1. Find the maclaurin series of $f(z) = e^z$

A: Maclaurin series is $f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \ldots$ — ①

given $f(z) = e^z$, $f(0) = e^0 = 1$

$f'(z) = e^z$, $f'(0) = e^0 = 1$

$f''(z) = e^z$, $f''(0) = e^0 = 1$

$f'''(z) = e^z$, $f'''(0) = e^0 = 1$

— — — — — — — — —

Sub in equ ① ⇒

$$f(z) = 1 + \frac{z}{1!} \cdot 1 + \frac{z^2}{2!} \cdot 1 + \frac{z^3}{3!} \cdot 1 + \ldots$$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

2. $f(z) = \cos z$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} (f''(0)) + \frac{z^3}{3!} f'''(0) + \ldots \quad ①$$

given $f(z) = \cos z$, $f(0) = \cos 0 = 1$

$f'(z) = -\sin z$, $f'(0) = -\sin 0 = 0$

$f''(z) = -\cos z$, $f''(0) = -\cos 0 = -1$

$f'''(z) = \sin z$, $f'''(0) = \sin 0 = 0$

sub in equ ① ⇒

$$f(z) = 1 + \frac{z}{1!} \cdot 0 + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(0) + \ldots$$

$$= 1 + 0 - \frac{z^2}{2!} + 0 + \ldots$$

$$= 1 - \frac{z^2}{2!} + \ldots$$

3. $f(z) = \sin z$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) \quad \text{—①}$$

$f(z) = \sin z$, $f(0) = \sin 0 = 0$

$f'(z) = \cos z$, $f'(0) = \cos 0 = 1$

$f''(z) = -\sin z$, $f''(0) = -\sin 0 = 0$

$f'''(z) = -\cos z$ $f'''(0) = -\cos 0 = -1$

sub in equ ① ⇒

$$f(z) = 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(0) + \frac{z^3}{3!}(-1) + \ldots$$

$$= \frac{z}{1!} + 0 - \frac{z^3}{3!} + \ldots$$

$$= \frac{z}{1!} - \frac{z^3}{3!} + \ldots$$

Fourier series

If $f(x)$ be a periodic function with peri[od] $2l$, then the fourier series of $f(x)$ in the interval $[c, c+2l]$ is defined by

Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$ where

where $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Odd & Even function

A fn. $f(x)$ is said to be odd if $f(-x) = -f($

eg: $f(x) = x$,

$f(x) = x^3$

$f(x) = x^5$

$\boxed{f(x) = \sin x}$ etc...

A fn. $f(x)$ is said to be even if $f(-x) = f(x)$

eg: $f(x) = x^2$

$f(x) = x^4$

$f(x) = x^6$

$\boxed{f(x) = \cos x}$ etc..

$f(x) = \ldots$

Note:

① even × even = even $+ \times + = +$

② odd × odd = even $- \times - = +$

③ odd × even = odd $- \times + = -$

④ If a fn. $f(x)$ is even, then $\boxed{b_n = 0}$

⑤ If $f(x)$ is odd, then $\boxed{a_n = 0}$

1. Develop the fourier series of $f(x) = x$, where $-\pi < x < \pi$

fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$$

given interval $= -\pi < x < \pi$

length = upper bound − lower bound

$= \pi - (-\pi) = \pi + \pi$

$= 2\pi$

$\therefore l = \frac{\text{length}}{2} = \frac{\cancel{2}\pi}{\cancel{2}} = \pi$

$$a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x) \cancel{\cos \frac{n\pi x}{l}} \, dx$$

~~ans~~

Sub. value of l

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\cancel{\pi} x}{\cancel{\pi}} + b_n \sin \frac{n\cancel{\pi} x}{\cancel{\pi}} \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right] \quad \text{—①}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{\cancel{2}}\right] - \left[\frac{(-\pi)^2}{}\right]$$

$$= \frac{1}{2\pi}\left[\pi^2 - \pi^2\right]$$

$$= \frac{1}{2\pi} \times 0 = 0$$

Result

$$\int_{-a}^{a} f(x)\,dx = 0, \text{ (o)}$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ (e)}$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos\frac{n\cancel{\pi}x}{\cancel{\pi}}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x}_{\text{odd}} \underbrace{\cos nx}_{\text{even}}\,dx$$

odd × even → odd → 0

$$= 0$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\frac{\pi n x}{l}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin\frac{n\cancel{\pi}x}{\cancel{\pi}}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x}_{\text{odd}} \underbrace{\sin nx}_{\text{odd}}\,dx$$

⇒ even

$$= 2\int_{0}^{\pi} f(x)\,dx = \frac{1}{\pi}\,2\int_{0}^{\pi} x \sin nx\,dx$$

$$= \cancel{\frac{2}{\pi}\int_0} = \frac{2}{\pi}\int_{0}^{\pi} \underbrace{x}_{1st} \underbrace{\sin nx}_{2nd}\,dx$$

modified integration by parts $\Rightarrow$ 1st $\int$ 2nd $-\frac{d}{dx}$(1st) $\cdot \int$ 2nd $+ \frac{d}{dx}$ (1st) $\int$ 2n

$$= \frac{2}{\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right) + 0\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\pi\cdot\left(\frac{-\cos n\pi}{n}\right) - 1\left(\frac{-\overset{\to 0}{\sin n\pi}}{n^2}\right) + 0\right] - \left[0\cdot\left(\frac{-\cos n0}{n}\right) - 1\cdot\left(\frac{-\sin n0}{n^2}\right) + 0\right]\right.$$

$\cos 0 =$

$\sin 0 = 0.$

$$= \frac{2}{\pi}\left\{\left[\pi\cdot\left(\frac{-\cos n\pi}{n}\right) + \left(\frac{\sin n\pi}{n^2}\right)\right] - 0\right\}$$

$$= \frac{2}{\cancel{\pi}} \cdot \frac{-\cancel{\pi}\cos n\pi}{n} = \frac{-2\cos n\pi}{n}$$

sub in ①

$$f(x) = 0 + \sum_{n=1}^{\infty}\left[0 + \frac{-2}{n}\cos n\pi \sin nx\right]$$

$$= \sum_{n=1}^{\infty} \frac{-2}{n}\cos n\pi \sin nx$$

2. Find the fourier series of $f(x) = |x|$ in $[-\pi, \pi]$

A: length $= \pi - (-\pi)$

$= \pi + \pi = 2\pi$

$\therefore\ l = \frac{\text{length}}{2} = \frac{\cancel{2}\pi}{\cancel{2}} = \pi$

$\therefore$ Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\cancel{\pi}x}{\cancel{\pi}} + b_n \sin\frac{n\cancel{\pi}x}{\cancel{\pi}}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \longrightarrow ①$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|}_{\text{even}}\, dx$$

$$= \frac{1}{\pi}\cdot 2\int_{0}^{\pi} |x|\, dx$$

$0 - \pi \Rightarrow$ full positive

$\because |x| = \begin{cases} -x \\ x \end{cases}$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\, dx$$

{ in 0 to π, $|x|$ takes +ve values only

$$= \frac{\cancel{2}}{\pi}\left[\frac{x^2}{\cancel{2}}\right]_0^{\pi}$$

$$= \frac{\cancel{2}1}{\cancel{\pi}}\left[\pi^{\cancel{2}} - 0\right] = \underline{\underline{\pi}}$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos\frac{n\cancel{\pi} x}{\cancel{\pi}}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|}_{\text{even}} \times \underbrace{\cos nx}_{\text{even}}\, dx \Rightarrow \text{even}$$

$$= \frac{1}{\pi}\, 2\int_{0}^{\pi} |x| \cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\left(\frac{-\cos nx}{n^2}\right) + 0\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\pi\cdot\frac{\cancelto{0}{\sin n\pi}}{n} - 1\cdot\frac{\cos n\pi}{n^2} + 0\right] - \left[0\cdot \cancelto{0}{\sin 0} + \frac{\cancelto{1}{\cos 0}}{n^2}\right]\right\}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{\pi}\int_c^{c+2l} |x| \sin \frac{n\pi x}{\pi}\, dx$$

$$= \frac{1}{\pi}\int_c^{c+2l} \underset{\text{even}}{|x|}\ \sin \underset{\text{odd}}{nx}\, dx$$

even $\cdot$ odd $\Rightarrow$ odd $= 0$

$$= \underline{\underline{0}}$$

sub in equ ① $\Rightarrow$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2\cancel{\pi}}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right] + 0 \right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{\pi} \frac{\cos n\pi - 1}{n^2} \right]$$

Half Range Expansions

Cosine Series — Sine Series

Half Range Cosine / Cos series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$\boxed{l = \text{length}}$

where $a_0 = \frac{2}{l}\int_0^l F(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l}\,dx.$$

Half range sine series

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx$

1. Find the half range cosine series for $F(x) = x^2$ in $0 \le x \le \pi$

Here l = length $= \pi - 0 = \pi$

$\therefore$ cosine series is $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} (1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi$$

Scanned with OKEN Scanner

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$= \frac{2}{\cancel{\pi}} \cdot \frac{\pi^{\cancel{3}}}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2 \cos\frac{n\cancel{\pi} x}{\cancel{\pi}}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \underbrace{x^2}_{1}\,\underbrace{\cos nx}_{2}\,dx$$

$$= \frac{2}{\pi}\left[x^2 \underbrace{\frac{\sin nx}{n}}_{\int 2nd} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right) - 0\right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\left[\pi^2 \frac{\sin n\pi \rightarrow 0}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi \rightarrow 0}{n^3}\right] - \left[\underbrace{0^2\frac{\sin n0 \rightarrow 0}{n}}_{0} + \underbrace{2\cdot 0\frac{\cos n0 \rightarrow 1}{n^2}}_{0} - \underbrace{2\frac{\sin n0 \rightarrow 0}{3}}_{0}\right]\right\}$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{2\cancel{\pi}\cos n\pi}{n^2}\right] = \frac{4}{n^2}\cos n\pi \longrightarrow \cos n\pi = (-1)^n$$

$$= \frac{4}{n^2}(-1)^n$$

sub in equ①

$$f(x) = \frac{2\pi^2}{3}\Big/2 + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\right]\cos nx$$

2. obtain the fourier sine series for $f(x) = x$ in $0 < x < \pi$

A: Here $l = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin nx \quad \text{———} ①$$

Scanned with OKEN Scanner

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{x\cdot\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right) + 0\right\}_0^\pi$$

$$= \frac{2}{\pi}\left\{\left[\pi\left(-\frac{\cos n\pi}{n}\right) + \frac{\overset{0}{\sin n\pi}}{n^2}\right] - 0\cdot\left(\frac{-\cos n0}{n}\right) + \left(\frac{\overset{0}{\sin n0}}{n^2}\right)\right\}$$

$$= \frac{2}{\cancel{\pi}}\left[-\frac{\cancel{\pi}\cos n\pi}{n}\right]$$

$$= \frac{-2(-1)^n}{n}$$

Sub. in ①

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin nx$$

8) Find the Fourier sine series of $f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \end{cases}$

A: here the given interval is $0<x<2$

$\therefore l = 2$

Fourier sine series is $f(x) = \sum b_n \sin\frac{n\pi x}{l}$

ie, $f(x) = \sum b_n \sin\frac{n\pi x}{2}$ —— ①

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x \sin\frac{n\pi x}{2}\,dx \int_1^2 2-x \sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 \underbrace{x}_{1st} \underbrace{\sin \frac{n\pi x}{2}}_{2nd} dx + \int_1^2 \underbrace{(2-x)}_{1st} \underbrace{\sin \frac{n\pi x}{2}}_{2nd} dx$$

$$= \left\{ x \cdot \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - 1 \left(\frac{-\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) + 0 \right\}_0^1 +$$

$$\left\{ (2-x) \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - (-1) \left(\frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) + 0 \right\}_1^2$$

$$= \left\{ 1 \cdot \left(\frac{-\cos \frac{n\pi \cdot 1}{2}}{\frac{n\pi}{2}} \right) - 1 \left(\frac{-\sin \frac{n\pi \cdot 1}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) - 0 \right\} +$$

$$\left\{ \left[(2-2) \left(\frac{-\cos \frac{n\pi \cdot 2}{2}}{\frac{n\pi}{2}} \right) - (-1) \left(\frac{-\sin \frac{n\pi 2}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) \right] - \right.$$

$$\left. \left[(2-1) \left(\frac{-\cos \frac{n\pi 1}{2}}{\frac{n\pi}{2}} \right) - (-1) \left(\frac{-\sin \frac{n\pi 1}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) \right] \right\}$$

$$= \left\{ \frac{-\cos \frac{n\pi}{2}}{\frac{n\pi}{2}} + \frac{\sin \frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2} \right\} + \left\{ \left(0 - \frac{\sin \frac{n\pi 2}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) - \right.$$

$$\left. \left(\frac{-\cos \frac{n\pi}{2}}{\frac{n\pi}{2}} - \frac{\sin \frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) \right\}$$

$$= \left\{ \frac{\cancel{-}2}{n\pi} \cancel{\cos \frac{n\pi}{2}} + \left(\frac{2}{n\pi}\right)^2 \sin \frac{n\pi}{2} \right\} - \left(\frac{2}{n\pi}\right)^2 \sin n\pi$$

$$+ \frac{\cancel{2}}{n\pi} \cancel{\cos \frac{n\pi}{2}} + \left(\frac{2}{n\pi}\right)^2 \sin \frac{n\pi}{2} \quad 0$$

$$= 2\left[\left(\frac{2}{n\pi}\right)^2 \sin \frac{n\pi}{2}\right] = \frac{8}{(n\pi)^2} \sin \frac{n\pi}{2}$$