Module 1 Syllabus Discription (9 Hrs)

Linear System of equations: Gauss elimination, Row echelon form, Linear independance: Rank of a matrix, Solution of linear System: Existence, uniqueness (without proof). The matrix eigen value problem, Determining eigen values and Eigen vectors. Diagonalization of matrices.

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Applications:-

1) Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 books, while Nadeem needs 8 pen and 10 books. They both go to a shop to enquire about the rates which are quoted as follows:- Pen 5/- each and book 50/- each. How much money does each need to spend?

Clearly meera needs Rs $(5\times2 + 50\times5)$, that is Rs 260 while Nadeem needs $(8\times5 + 50\times10)$, that is Rs. 540.

In terms of matrix representation, we can write the above information as follows:-

Requirements | Price / Peace | Money needed

$$\begin{array}{cc} \begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \\ \downarrow \quad \downarrow \\ \text{Pen} \quad \text{book} \end{array} \quad \begin{bmatrix} 5 \\ 50 \end{bmatrix} \quad \begin{bmatrix} (2\times5)+(5\times50) \\ (8\times5)+(10\times50) \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose they ~~eq~~ enquire about ~~that~~ the rates from another shop, quoted as follows:-

Pen - 4/- each and book 40/- each.

Again this information can be represented as

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 40 \end{bmatrix} \quad \begin{bmatrix} 2\times4 + 5\times40 \\ 8\times4 + 10\times40 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$$

Now, the information in both the cases can be combined and expressed in terms of matrices as follows:-

$$\underset{\text{Requirements}}{\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix}} \quad \underset{\text{Price per piece}}{\begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix}} \quad \underset{\text{Money needed}}{\begin{bmatrix} (2\times5)+(5\times50) & (2\times4)+(5\times40) \\ (8\times5)+(10\times50) & (8\times4)+(10\times40) \end{bmatrix}}$$

$$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

2) Image processing:- Matrices are used to apply filters, performs transformations, and compress images.

3) Medical imaging:- Matrices reconstruct images from data in MRI and CT scans.

4) Traffic flow Modeling:- Matrices optimize traffic ~~flow~~ light timing and traffic flow.

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Linear Independence: Rank of a Matrix

Row Echelon form of a matrix:

First non-zero element of a row from the left is called leading element or pivot of the row.

A matrix is said to be in its row echelon form or row reduced echelon form, if the leading element in each row (if exist) is 1 and the number of zeros before the leading element in each row is greater than the corresponding number of zeros of the preceeding rows.

Eg:- $A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 5 & 0 & 3 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

↓
Row reduced form.
(If the leading element is not 1.)

Rank of a Matrix:-

The rank of a matrix is equal to the number of non-zero rows in its equivalent row reduced form (or row reduced echelon form).

Que 1:- Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & 4 & 1 \\ 5 & 6 & 7 & 5 \end{bmatrix}$

Ans:- Given $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & 4 & 1 \\ 5 & 6 & 7 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 6 & -5 \\ 5 & 6 & 7 & 5 \end{bmatrix} \qquad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 6 & -5 \\ 0 & -4 & 12 & -10 \end{bmatrix} \qquad R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 6 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_2 - 2R_2$$

Rank = no. of non-zero rows $= \underline{\underline{2}}$.

Que 2:- Reduce to row echelon form and find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \qquad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 8 & 1 & 14 & 17 \end{bmatrix} \qquad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \qquad R_4 \to R_4 - 8R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & -10 & -15 \end{bmatrix} \qquad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to R_4 - 5R_2$$

Rank = 2

Que 3:- Find the rank of the matrix $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Ans:-

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 21 & -21 & 0 & -15 \end{bmatrix} \qquad R_2 \to R_2 + 2R_1$$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \qquad R_3 \to R_3 - 7R_1$$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to 2R_3 + R_2$$

$\therefore$ Rank = 2

Que 4:- Find the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Ans:-

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \qquad \cancel{R_2 \to 2R_2 - R_1} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 4 & 9 & 1 \\ 0 & 9 & 12 & -1 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$$

-2+3

-1+6

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & -33 & -1 \\ 0 & 9 & 12 & -1 \end{bmatrix} \qquad R_3 \to 4R_2 - 5R_3$$

12 - 45

4 - 5

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & -33 & -1 \\ 0 & 0 & 33 & -14 \end{bmatrix} \qquad R_4 \to 5R_4 - 9R_2$$

60 − 27 = 33

−5−9

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & -33 & -1 \\ 0 & 0 & 0 & -15 \end{bmatrix} \qquad R_4 \to R_4 + R_3$$

$\therefore$ Rank $= 4$

Que5:- Determine the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$

Ans:-

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \qquad R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad R_3 \to R_3 - R_2$$

$\therefore$ Rank $= 3$

Que6:- Find the rank of $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - 4R_2$$

Rank $= 2$

Que:- **Exercise:-**

(1) Find the rank of the following matrices:-

(1) $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ Rank = 2

(2) $\begin{bmatrix} -2 & 4 & -6 \\ 1 & -2 & 3 \end{bmatrix}$ Rank = 1.

(3) $\begin{bmatrix} 0 & 0 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ Rank = 3

(4) $\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 8 \\ 2 & 0 & 4 \end{bmatrix}$ Rank = 3

(5) $\begin{bmatrix} 6 & 0 & -3 & 0 \\ 0 & -1 & 0 & 5 \\ 2 & 0 & -1 & 0 \end{bmatrix}$ Rank = 2.

Rank of a Matrix and Linearly independence

Vectors:- A vector is a matrix with only one row or column. Its entries are called the components of the vectors.

A Set of 'n' vectors $\{v_1, v_2, \dots, v_n\}$ are linearly independent if the rank of the matrix A is equal to the number of rows of the matrix, where A is a matrix obtained by taking each vector as a row.

The vectors are linearly dependent if the rank of A is less than the number of rows of A.

Que 7:- Are the vectors $(3,-1,4)$, $(6,7,5)$ and $(9,6,9)$ linearly dependent or independent?

Ans:- $A = \begin{bmatrix} 3 & -1 & 4 \\ 6 & 7 & 5 \\ 9 & 6 & 9 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 9 & -3 \\ 9 & 6 & 9 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 9 & -3 \\ 0 & 9 & -3 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

Here rank of $A = 2 <$ the no. of rows of A

$\Rightarrow$ Given vectors are linearly dependent.

Que 8:- Check whether the vectors $(1, 2, -1, 3)$, $(2, -1, 3, 2)$ and $(-1, 8, -9, 5)$ are linearly independent or not?

Ans:-
$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 2 \\ -1 & 8 & -9 & 5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -4 \\ -1 & 8 & -9 & 5 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_1$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -4 \\ 0 & 10 & -10 & 8 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here rank $= 2 <$ the no. of rows of A.

$\therefore$ The given vectors are linearly dependent.

Que 9:- Show that the vectors $(3, 4, 0, 1)$, $(2, -1, 3, 5)$ and $(1, 6, -8, -2)$ are linearly independent.

Ans:-
$$A = \begin{bmatrix} 3 & 4 & 0 & 1 \\ 2 & -1 & 3 & 5 \\ 1 & 6 & -8 & -2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$
$$\sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 2 & -1 & 3 & 5 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$
$$\sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 0 & -13 & 19 & 9 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$
$$\sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 0 & -13 & 19 & 9 \\ 0 & -14 & 24 & 7 \end{bmatrix}$$

4-18

1+6

$R_3 \rightarrow 13R_3 - 14R_2$
$$\sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 0 & -13 & 19 & 9 \\ 0 & 0 & 46 & -35 \end{bmatrix}$$

Here Rank $= 3 =$ no. of rows of A.

$\therefore$ The given vectors are linearly independent.

Que 10:- Check whether the vectors $(1, 2, 1)$, $(2, 1, 4)$, $(4, 5, 6)$ and $(1, 8, -3)$ are linearly dependent or not.

Ans:-
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 4 & 5 & 6 \\ 1 & 8 & -3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

$R_4 \rightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 6 & -4 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $= 3 <$ the no. of rows of A.

$\therefore$ The given vectors are linearly dependent.

Exercise

Check whether the following vectors are linearly dependent or not :-

1) $(3,4,0,1)$, $(2,-1,3,5)$ and $(1,6,-8,-2)$ No.

2) $(1,0,1)$, $(1,1,0)$, $(0,1,0)$ Yes

3) $(2,0,0,7)$, $(2,0,0,8)$, $(2,0,0,9)$ and $(2,0,1,0)$ No.

4) $(9,8,7,6,5)$ and $(9,7,5,3,1)$ Yes

5) $(6,0,-1,3)$, $(2,2,5,0)$ and $(-4,-4,-4,-4)$ Yes

System of Linear Equations

We consider the following system of m linear equations in n unknowns $x_1, x_2, x_3, \dots, x_n$:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \longrightarrow (1)$$

Changing to matrix notation, these equations can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Or $AX = B$

Here A is called the coefficient matrix of the system of equations.

The matrix $[AB]$ is defined by

$$[AB] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

is called the augmented matrix of the system of equation (1)

In the system $AX = B$, if $B = 0$, then it is called a homogeneous system of linear equations and if $B \neq 0$, then it is called a non-homogeneous system of linear equations.

Consistant System of equations :- The system of equations $AX = B$ is consistant if and only if the coefficient matrix A and the augmented matrix $[AB]$ are of the same rank. Otherwise it is inconsistant

Note:- If the system is consistant, then the system of equations $AX=B$ having one or more solutions. Otherwise it has no solutions.

Gauss Elimination Method

The process of reducing a given matrix to echelon form using a chain of elementary row operations is called Gauss elimination method. Using this process reduce the given augmented matrix $[AB]$ to the echelon form. Then there arise the following cases.

Case 1:- If $\text{rank}(A) \neq \text{rank}(AB)$, then the system is inconsistant and has no solutions.

Case 2:- If $\text{rank}(A) = \text{rank}(AB) =$ the number of unknowns, then the system is consistant and has a unique sol^n.

Case 3:- If $\text{rank}(A) = \text{rank}(AB) <$ no. of unknowns, then the system is consistent and has infinite number of solutions.

If the system $AX=B$ is consistent, then the solution can be obtained using back substitution.

Que 1:- Solve by Gauss Elimination method:-

$$x - y + z = 0$$
$$-x + y - z = 0$$
$$10y + 25z = 90$$
$$20x + 10y = 80.$$

Ans:- The Given System can be written in the form $AX=B$ as

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

The augmented matrix $[AB]$ is

$$[AB] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 20R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \end{bmatrix}$$

$$[AB] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$[AB] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$[AB] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 20 & 10 & 0 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 20R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 3R_3 - R_2$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95 & 190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here Rank $(A) = 3$

and rank $(AB) = 3$.

$\Rightarrow$ rank(A) = rank (AB)

= the no. of unknowns.

then the System has unique Solution.

The System can be written as

$AX = B$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 30 & -20 \\ 0 & 0 & 95 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 190 \\ 0 \end{bmatrix}$$

$\Rightarrow x - y + z = 0$.

$30y - 20z = 80$

$95z = 190$

$\Rightarrow z = \frac{190}{95} = 2$.

$30y - (20 \times 2) = 80$

$30y = 120$

$\Rightarrow y = \frac{120}{30} = 4$.

$x - 4 + 2 = 0$

$x - 2 = 0 \Rightarrow x = 2$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

Que 2:- Solve the following system by Gauss - Elimination method:-

$x + y + z = 6$

$x + 2y - 3z = -4$

$-x - 4y + 9z = 18$.

Ans:- $AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ -1 & -4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 18 \end{bmatrix}$$

Augmented matrix

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & -3 & -4 \\ -1 & -4 & 9 & 18 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ -1 & -4 & 9 & 18 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -3 & 10 & 24 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -2 & -6 \end{array}\right)$$

Here rank (A) = 3

= rank (AB) = no. of unknowns.

Then the system is consistant and has unique solution.

$AX = B$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ -6 \end{pmatrix}$$

$\Rightarrow x + y + z = 6.$

$y - 4z = -10$

$-2z = -6$

$\Rightarrow \underline{\underline{z = 3}}$

$y - (4 \times 3) = -10$

$y = -10 + 12$

$\Rightarrow \underline{\underline{y = 2}}$

$x + 2 + 3 = 6.$

$\Rightarrow \underline{\underline{x = 1}}$

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Que 8:- Using Gauss elimination solve the following system of equations:-

$x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

$x - y + z = -1$

Ans:- $AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

The augmented matrix is

$$[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_3 \to 7R_3 - 6R_2$

$R_4 \to 7R_4 - 3R_2$

35 − 30
− 28 + 48
14 − 15

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$R_4 \to 5R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 3 = Rank (AB)

= no. of unknowns

∴ The system has unique solution:-

$AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 20 \end{bmatrix}$$

$\Rightarrow x + 2y - z = 3$

$-7y + 5z = -8$

$5z = 20$

$\Rightarrow z = 4.$

$-7y + (5 \times 4) = -8$

$\therefore -7y = -8 - 20$

$-7y = -28 \Rightarrow \underline{y = 4}$

$x + (2 \times 4) - 4 = 3.$

$x + 8 - 4 = 3$

$x + 4 = 3$

$\Rightarrow \underline{x = -1}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$$

Que 4:- Solve the given system using Gauss Elimination method:-

$3x + 3y + 2z = 1$

$x + 2y = 4$

$10y + 3z = -2$

$2x - 3y - z = 5$

Ans:- $AX = B$

$$= \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix:-

$$(AB) = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$R_3 \to 3R_3 + 10R_2$

$R_4 \to 3R_4 - 7R_2.$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & -17 & 68 \end{bmatrix}$$

$R_4 \to 29R_4 + 17R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{Rank}(A) = 3 = \text{Rank}(AB)$

= no. of unknowns.

∴ The solution is unique.

$AX = B$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 29 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -116 \\ 0 \end{bmatrix}$$

$x+2y=4$

$-3y+2z=-11$

$29z=116$

$\Rightarrow z = \frac{116}{29} = -4$

$-3y+(2\times -4) = -11$

$-3y-8=-11$

$-3y = -3 \Rightarrow y=1$

$x+(2\times 1)=4$

$x+2=4 \Rightarrow x=2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

$\boxed{\text{Que5:-}}$ Examine the consistency for the following equations:-

1) $x+y+2z=2$

$2x-y+3z=2$

$5x-y+8z=10.$

Ans:- $AX=B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}$$

Augmented matrix

$$[AB] = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 8 & 10 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

$\text{Rank}(A) = 2$

$\text{Rank}(AB) = 3$

$\therefore \text{Rank}(A) \neq \text{rank}(AB)$

$\therefore$ The system is inconsistant.

2) $4y+3z=8$

$2x-z=2$

$3x+2y=5$

Ans:- $AX=B$

$$\begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

$$[AB] = \begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix}$$

$R_3 \to 3R_1 - 2R_3$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & -4 & -3 & -4 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\text{Rank}(A) = 2$

$\text{Rank}(AB) = 3.$

$\text{Rank}(A) \neq \text{Rank}(AB).$

The system is inconsistant

Que 6:- Solve the System of eq^{n}s by Gauss Elimination :-

$$4y + 4z = 24$$
$$3x - 11y - 2z = -6$$
$$6x - 17y + z = 18$$

Ans:- $AX = B$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 4 \\ 3 & -11 & -2 \\ 6 & -17 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ -6 \\ 18 \end{bmatrix}$$

Augmented matrix

$$(A B) = \begin{bmatrix} 0 & 4 & 4 & 24 \\ 3 & -11 & -2 & -6 \\ 6 & -17 & 1 & 18 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 3 & -11 & -2 & -6 \\ 0 & 4 & 4 & 24 \\ 6 & -17 & 1 & 18 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 3 & -11 & -2 & -6 \\ 0 & 4 & 4 & 24 \\ 0 & 5 & 5 & 30 \end{bmatrix}$$

$R_3 \to 5R_2 - 4R_3$

$$\sim \begin{bmatrix} 3 & -11 & -2 & -6 \\ 0 & 4 & 4 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here rank$(A) = 2 =$ rank(AB) $<$ no. of unknowns.

$\therefore$ The System has infinite no. of sol^{n}s.

$AX = B$

$$\begin{bmatrix} 3 & -11 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 24 \\ 0 \end{bmatrix}$$

Free variable $= n - r = 3 - 2 = 1$

$$\Rightarrow 3x - 11y - 2z = -6$$
$$4y + 4z = 24$$

Let $z = t$

$$\Rightarrow 4y + 4t = 24$$
$$4y = 24 - 4t$$
$$y = 6 - t$$

$$3x - 11(6 - t) - 2t = -6$$
$$3x - 66 + 11t - 2t = -6$$
$$3x = 60 + (-9t)$$
$$\Rightarrow x = 20 - 3t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 - 3t \\ 6 - t \\ t \end{bmatrix}$$

Que 7:-
$$y + z - 2w = 0$$
$$2x - 3y - 3z + 6w = 2$$
$$4x + y + z - 2w = 4$$

Ans:- $AX = B$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 & -2 \\ 2 & -3 & -3 & 6 \\ 4 & 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$[AB] = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$R_3 \to 2R_1 - R_3$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -7 & -7 & 14 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 7R_2$.

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank (A) = 2 = rank (AB)

< the no of unknowns = 4

∴ Free variables = 4 − 2 = 2.

$AX = B$

$$\begin{bmatrix} 2 & -3 & -3 & 6 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x - 3y - 3z + 6\omega = 2$

$y + z - 2\omega = 0$

Let $\omega = t$, $z = s$

$\Rightarrow y + s - 2t = 0$

$\underline{y = 2t - s}$.

$2x - 3(2t - s) - 3s + 6t = 2$.

$2x - 6t + 3s - 3s + 6t = 2$

$2x = 2 \Rightarrow \underline{x = 1}$

$$\begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 2t - s \\ s \\ t \end{bmatrix}$$

Que 8:- Find the values of μ ~~and~~ for which the system of eqn

$x + y + z = 1$
$x + 2y + 3z = \mu$
$x + 5y + 9z = \mu^2$

is consistant. For each value of μ obtained, find the solution of the system.

Ans:- $AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \mu \\ \mu^2 \end{bmatrix}$$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu - 1 \\ 0 & 4 & 8 & \mu^2 - 1 \end{bmatrix}$$

$R_4 \to R_4 - 4R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu - 1 \\ 0 & 0 & 0 & \mu^2 - 4\mu + 3 \end{bmatrix}$$

Given the system is consistant

$\Rightarrow$ rank (A) = rank (AB)

Here Rank(A) = 2

Then rank (AB) = 2 is possible only if $\mu^2 - 4\mu + 3 = 0$.

$(\mu - 3)(\mu - 1) = 0$

$\Rightarrow \mu = 1, 3$.

If $\mu = 1$

$AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Rank (A) = Rank (AB) < no. of unknowns

free variable = 3 − 2 = 1

$$x+y+z=1$$
$$y+2z=0$$

Let $z=t$

$$y+2t=0 \Rightarrow \underline{y=-2t}$$

$$x-2t+t=1$$
$$x-t=1 \Rightarrow$$
$$\underline{x=1+t}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ -2t \\ t \end{bmatrix}$$

When $\mu=3$. $AX=B \Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$x+y+z=1$$
$$y+2z=2$$

Let $z=t$

$$y+2t=2$$
$$\Rightarrow y=\underline{2-2t}$$
$$x+(2-2t)+t=1$$
$$x+2-t=1$$
$$\underline{x=t-1}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t-1 \\ 2-2t \\ t \end{bmatrix}$$

Que 9:- Find the value of λ and μ for which the system of equations :-

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu \text{ has}$$

(i) no solution.

(ii) Unique solution.

(iii) Infinite solution.

Ans:- $AX=B$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$[AB] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$R_2 \to 7R_1 - 2R_2, \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

(i) no Solution :-

$\Rightarrow$ rank$(A) \neq$ ran(AB).

It is possible only if

$$\lambda-5=0 \quad \text{and} \quad \mu-9\neq 0$$
$$\Rightarrow \lambda=5 \quad \text{and} \quad \mu\neq 9$$

(ii) Unique solution:-

Rank (A) = rank(AB) = no. of unknowns $=3$

It is possible only if

$\lambda-5\neq 0$ and $\mu-9$ take any value

$\Rightarrow \lambda\neq 5$ and μ can take any value.

(iii) Infinite Solution

$Rank(A) = rank(B) < $ no of unknowns $= 3$

It is possible only when

$Rank(A) = 2 = Rank(AB)$

$\Rightarrow \lambda - 5 = 0$ and $\mu - 9 = 0$.

$\Rightarrow \lambda = 5$ and $\mu = 9$

Excercise.

Que:- (I) By Using Gauss Elimination Solve the following system of equations :-

1) $x + 2y - z = 0$
$3x + y - z = 0$
$2x - y = 0$.

2) $8y + 6z = -4$
$-2x + 4y - 6z = -4$
$x + y - z = 2$.

3) $10x + 4y - 2z = 14$
$-3w - 15x + y + 2z = 0$
$w + x + y = 6$
$8w - 5x + 5y - 10z = 26$.

(II) Find the values of a and b for which the system of linear equations

$x + 2y + 3z = 6$
$x + 3y + 5z = 9$
$2x + 5y + az = b$ has

(i) no Solution.
(ii) Unique Solution
(iii) Infinite no. of solutions.

Que 10:- Solve the system

$x - y + z = -2$
$2x + y - z = 5$
$3x - 2y + 2z = -3$.

Ans:- $AX = B$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}$$

$$AB = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2 & 1 & -1 & 5 \\ 3 & -2 & 2 & -3 \end{array}\right] \quad \begin{matrix} R_2 \to 2R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \end{matrix}$$

$-3+1$
$3+2$
$-6+3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -3 & 3 & -9 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -3 & 3 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to 3R_3 - R_2$$

$Rank(A) = Rank(AB) <$ no. of unknows.

$\Rightarrow$ Consistant and infinite no. of solutions

Free variable $= n - r = 3 - 2 = 1$

$x - y + z = -2 \longrightarrow (1)$

$-3y + 3z = -9 \longrightarrow (2)$

Let $z = t$. from (2),

$-3y + 3t = -9$

$3[-y + t] = -9$

$\Rightarrow -y + t = -3 \Rightarrow -y = -3 - t$.

$\Rightarrow y = 3 + t$.

from (1)

$x - (3 + t) + t = -2$.

$x - 3 - t + t = -2$

$x - 3 = -2 \Rightarrow x = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3+t \\ t \end{bmatrix}$$

Homogeneous System ?

Eigen Values and Eigen Vectors.

A matrix eigenvalue problem consider the vector equation $AX = \lambda X \longrightarrow (1)$.

Here A is a given square matrix, λ an unknown scalar and x an unknown vector. In a matrix eigen value problem, the task is to determine λ's and x's that satisfies (1).

The λ's satisfy (1) are called the eigen values of A and the corresponding non-zero x's that also satisfy (1) are called eigen vectors of A. The problem for finding non zero x's and λ's that satisfies equation (1) is called an eigenvalue problem.

The equation (1) can be written as:

$$Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0.$$

This is a system of linear homogeneous equation and has no trivial solution ($x \neq 0$) only if $|A - \lambda I| = 0$.

Result:- If the homogeneous system of equation $Ax=0$ has non-trivial solution, then $|A|=0$.

The equation $|A - \lambda I| = 0$ is called the characteristic equation (or latent equation).

The roots of the characteristic equation are the eigen values (latent values or characteristic value).

The solution $x \neq 0$ are called the eigen vectors or characteristic vectors of A corresponding to that eigen value λ.

The set of all eigen values of A is called the Spectrum of A

Note:- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix

The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - (a_{11} + a_{22}) \cdot \lambda + |A| = 0$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3x3 matrix.

The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\}\lambda - |A| = 0.$$

$$\left[\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\}\lambda - |A| = 0. \right]$$

Que 1:- Find all eigen values of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

Ans:- The characteristic equation is given by $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - (a_{11} + a_{22})\lambda + |A| = 0.$ $\qquad |A| = -2 - 4 = -6.$

$\Rightarrow \lambda^2 - (1 + (-2))\lambda + \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda - 2)(\lambda + 3) = 0 \Rightarrow \lambda = 2, -3.$

$\therefore$ The eigen values are 2 and −3.

Que 2:- Find the eigen values and eigen vectors of the matrix

$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Ans:- Ch. eq^n : $\lambda^2 - (a_{11} + a_{22})\lambda + |A| = 0$

$\Rightarrow \lambda^2 - (8 + 2)\lambda + \begin{vmatrix} 8 & -4 \\ 2 & 2 \end{vmatrix} = 0.$ $\qquad \begin{vmatrix} 8 & -4 \\ 2 & 2 \end{vmatrix} = 16 + 8$

$\Rightarrow \lambda^2 - 10\lambda + 24 = 0$

$\Rightarrow (\lambda - 4)(\lambda - 6) = 0$ $\qquad \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \lambda = 4, \lambda = 6$

$\therefore$ the eigen values are 4 and 6.

When $\lambda = 4$

The given equation is

$(A - 4I)X = 0$

$$\left\{\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x - 4y = 0$

$2x - 2y = 0$

$\Rightarrow x - y = 0$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$$\frac{x}{-1} = \frac{-y}{+1} \Rightarrow \frac{x}{1} = \frac{y}{1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $\lambda = 6$

$(A - 6I)X = 0$

$$\Rightarrow \left\{\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right\}X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x - 4y = 0$

$2x - 4y = 0$

$\Rightarrow x - 2y = 0$

$x - 2y = 0$

$\Rightarrow x = 2y$

$$\frac{x}{-2} = \frac{-y}{+1} \Rightarrow \frac{x}{2} = \frac{y}{1}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vectors $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

Que 3:- Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Ans:- The characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\}\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - (1+2+3)\lambda^2 + \left\{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}\right\}\lambda - \begin{vmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + \{5 + 3 + 3\}\lambda - (1(6-1) - 1(-3-0) + 2(-1-0)) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

Put $\lambda = 1$, $1 - 6 + 11 - 6 = 0$

$\therefore \lambda = 1$ is a root.

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 2, 3$$

$$\therefore \lambda = 1, 2, 3$$

ie, the eigen values are 1, 2 and 3.

When $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$\Rightarrow \left\{\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 0x + y + 2z &= 0 \\ -x + y + z &= 0 \\ 0x + y + 2z &= 0 \end{aligned}\right\}$$

$$\frac{x}{1-2} = \frac{-y}{0--2} = \frac{z}{0--1}$$

$$\Rightarrow \frac{x}{-1} = \frac{-y}{2} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

When $\lambda = 2$

$$(A - \lambda I)X = 0$$

$$\Rightarrow \left\{\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -x + y + 2z &= 0 \\ -x + 0y + z &= 0 \\ 0x + y + z &= 0 \end{aligned}\right\}$$

$$\frac{x}{1-0} = \frac{-y}{-1--2} = \frac{z}{0--1}$$

$$\Rightarrow \frac{x}{1} = \frac{-y}{1} = \frac{z}{1}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-1}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{-1}$$

When $\lambda = 3$

$$(A - \lambda I)X = 0$$

$$\Rightarrow \left\{\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2x + y + 2z &= 0 \\ -x - y + z &= 0 \\ 0x + y + 0z &= 0 \end{aligned}\right\}$$

$$\frac{x}{0-1} = \frac{-y}{0-0} = \frac{z}{-1-0}$$

$$\frac{x}{-1} = \frac{-y}{0} = \frac{z}{-1}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vectors are

$$\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Que4:- Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Ans:- characteristic equation is:-

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + \left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\} \lambda - |A| = 0.$$

$$|A| = 1(6-2) - 0 + -1(2-4)$$

$$= 4 - -2 = 6$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + \left\{ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\} \lambda - 6 = 0.$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + (4+5+2)\lambda - 6 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

Put $\lambda = 1$.

$$1 - 6 + 11 - 6 = 0.$$

$\therefore \lambda = 1$ is a root

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	$\boxed{0}$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0.$$

$$\Rightarrow \lambda = 2, 3$$

$\therefore$ Eigen values are 1, 2, 3

When $\lambda = 1$

$$(A - \lambda I)x = 0$$

$$\Rightarrow (A - I)x = 0.$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{aligned} 0x + 0y - z &= 0 \\ x + y + z &= 0. \\ 2x + 2y + 2z &= 0. \end{aligned} \right\}$$

$$\Rightarrow \frac{x}{0--1} = \frac{-y}{0--1} = \frac{z}{\begin{vmatrix} 0 & -0 \\ 1 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{0}.$$

$$\Rightarrow x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

When $\lambda = 2$

$$(A - \lambda I)x = 0$$

$$\Rightarrow (A - 2I)x = 0.$$

$$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{aligned} -x + 0y - z &= 0 \\ x + 0y + z &= 0 \\ 2x + 2y + z &= 0. \end{aligned} \right\}$$

$$\frac{x}{0-2} = \frac{-y}{1-2} = \frac{z}{2-0}$$

$$\Rightarrow \frac{x}{-2} = \frac{-y}{-1} = \frac{z}{2}$$

$$\Rightarrow \frac{x}{-2} = \frac{y}{1} = \frac{z}{2}.$$

$$X_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

When $\lambda = 3$

$$(A - 3I)X = 0.$$

$$\left\{\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2x + 0y - z &= 0 \\ x - y + z &= 0 \\ 2x + 2y + 0z &= 0 \end{aligned}\right\}$$

$$\frac{x}{0-2} = \frac{-y}{0-2} = \frac{z}{2+2}.$$

$$\frac{x}{-2} = \frac{-y}{-2} = \frac{z}{4}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2}.$$

$$\Rightarrow X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Ques 5:- Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Ans:-

Characteristic eq^n :- $|A - \lambda I| = 0$.

$$\Rightarrow \lambda^3 - (-2+1+0)\lambda^2 + \left\{\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right\}\lambda - |A| = 0.$$

$$\Rightarrow \lambda^3 + \lambda^2 + (-12 + -3 + -6)\lambda - (-2(-12) - 2(-6) - 3(-3))$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$$

When $\lambda = 1$, $1 + 1 - 21 - 45 \neq 0$

$\lambda = 5$, $125 + 25 - 105 - 45 = 0$

$\therefore \lambda = 5$ is a root.

5	1	1	-21	-45
	0	5	30	45
	1	6	9	0

$$\Rightarrow \lambda^2 + 6\lambda + 9 = 0.$$

$$\Rightarrow \lambda = -3, -3$$

$\therefore$ Eigen values are $5, -3, -3$.

When $\lambda = 5$ $\quad (A - \lambda I)X = 0$

$$\Rightarrow (A - 5I)X = 0$$

$$\Rightarrow \left\{\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -7x + 2y - 3z &= 0 \\ 2x - 4y - 6z &= 0 \\ -x - 2y - 5z &= 0 \end{aligned}\right\}$$

$$\frac{x}{-12-12} = \frac{-y}{42+6} = \frac{z}{28-4}$$

$$\Rightarrow \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\Rightarrow \frac{x}{-1} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore x_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

When $\lambda = -3$. $(A+3I)X = 0$

$$\left\{\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2y - 3z = 0$

$2x + 4y - 6z = 0.$

$-x - 2y + 3z = 0.$

$\Rightarrow x + 2y - 3z = 0.$

$x + 2y - 3z = 0.$

$x + 2y - 3z = 0.$

Three equations are same

$\Rightarrow$ put $z = t_1$, $y = t_2$.

$\Rightarrow \cancel{t_1 + 2t_2 - 3z = 0}$

$\Rightarrow x + 2t_2 - 3t_1 = 0$

$\Rightarrow x = 3t_1 - 2t_2.$

$$\therefore X = \begin{bmatrix} 3t_1 - 2t_2 \\ t_2 \\ t_1 \end{bmatrix}$$

$$= \begin{bmatrix} 3t_1 \\ 0 \\ t_1 \end{bmatrix} + \begin{bmatrix} -2t_2 \\ t_2 \\ 0 \end{bmatrix}$$

$$= t_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Excercise

1) Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Find the eigen values and eigen vectors of the following matrices:

(i) $A = \begin{bmatrix} -1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(iv) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Diagonalization

→ **Properties of Eigen Values:**

1) Eigen values of the matrix A and A^T are same.
2) If λ is an eigen value of A, then λ^n is an eigen value of A^n.
3) If λ is an eigen value of A, then $k\lambda$ is an eigen value of kA.
4) If λ is an eigen value of A, then $1/\lambda$ is an eigen value of A^{-1}
5) Eigen values of diagonal matrices are its diagonal elements.
6) The sum of the eigen values of a matrix is the sum of its diagonal elements.
7) The product of the eigen values of a matrix is equal to its determinant value.

Similar Matrices: Two matrices A and B are said to be similar if there exist an invertible matrix P such that $P^{-1}AP = B$.

Diagonalization:- Diagonalization is a process by which the given matrix A is reduced to a diagonal matrix such that the eigen values are same and both are similar matrices.

ie, the matrix A is said to be diagonalizable if there is an invertible matrix P, called modal matrix, such that $\boxed{P^{-1}AP = D}$ where P is obtained from the eigen vectors of A and D is a diagonal matrix, called the spectral matrix of A.

The diagonal elements of D are called the spectral values of A.

Note:- Given $D = P^{-1}AP$

$$\Rightarrow A = PDP^{-1}$$

$\Rightarrow$ For any +ve integer m $\quad A^m = PD^mP^{-1}$

Que 1:- Diagonalize the matrix

$$A=\begin{bmatrix}6&0&0\\12&2&0\\21&-6&9\end{bmatrix}$$

Ans:- Steps:-

1) Find the eigen values of A using the characteristic equation $|A-\lambda I|=0$.

2) For each eigen values, find the eigen vectors from the vector equation $(A-\lambda I)X=0$.

3) For a 3x3 matrix, we get 3 eigen vectors say X_1, X_2 and X_3. From this consider the model matrix $P=[X_1 \ X_2 \ X_3]$.

4) Find a diagonal matrix D from the equation

$$\boxed{D=P^{-1}AP}$$

For finding the matrix P^{-1} consider the following steps:-

(i) find ~~minor~~ cofactor matrix of P

(ii) find adjoint matrix of P which is obtained by taking transform of the cofactor matrix of P.

(iii) Then $P^{-1}=\frac{1}{|P|}\cdot \text{Adj } P$.

Ans:- Ch. eqn $|A-\lambda I|=0$

$$\Rightarrow \lambda^3-(6+2+9)\lambda^2+\left\{\begin{vmatrix}2&0\\-6&9\end{vmatrix}+\begin{vmatrix}6&0\\21&9\end{vmatrix}+\begin{vmatrix}6&0\\12&2\end{vmatrix}\right\}\lambda-|A|=0.$$

$$\Rightarrow \lambda^3-17\lambda^2+(18+\cancel{30}^{54}+12)\lambda-(6(18-0))=0$$

$$\Rightarrow \lambda^3-17\lambda^2+84\lambda-108=0.$$

when $\lambda=1$, $1-17+84-108\neq 0$

when $\lambda=2 \Rightarrow 8-68+168-108=0$

$\therefore \lambda=2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -17 & 84 & -108\\ & 0 & 2 & -30 & 108\\ \hline & 1 & -15 & 54 & \boxed{0}\end{array}$$

$$\Rightarrow \lambda^2-15\lambda+54=0$$

$$\Rightarrow (\lambda-6)(\lambda-9)=0$$

$$\Rightarrow \lambda=6,9.$$

$\therefore$ Eigen values are 2, 6, 9.

when $\lambda=2$

$(A-2I)X=0$

$$\Rightarrow \left\{\begin{bmatrix}6&0&0\\12&2&0\\21&-6&9\end{bmatrix}-\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\right\}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}4&0&0\\12&0&0\\21&-6&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l}4x+0y+0z=0\\12x+0y+0z=0\\21x+-6y+7z=0\end{array}\right\}$$

$$\frac{x}{0}=\frac{-y}{84-0}=\frac{z}{-72-0}$$

$$\Rightarrow \frac{x}{0}=\frac{y}{-84}=\frac{z}{-72} \quad \div 3$$

$$\Rightarrow \frac{x}{0}=\frac{y}{-28}=\frac{z}{-24}$$

$$\Rightarrow \cancel{\frac{x}{0} = \frac{y}{28} = \frac{z}{24}}$$

$$\therefore \cancel{X_1 = \begin{bmatrix} 0 \\ 28 \\ 24 \end{bmatrix}}$$

$$\Rightarrow \frac{x}{0} = \frac{y}{7} = \frac{z}{6}$$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 7 \\ 6 \end{bmatrix}$$

When $\lambda = 6$ $\quad (A - 6I)x = 0$

$$\Rightarrow \left\{\begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 12 & -4 & 0 \\ 21 & -6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 0x + 0y + 0z &= 0 \\ 12x - 4y + 0z &= 0 \\ 21x - 6y + 3z &= 0 \end{aligned}\right\}$$

$$\Rightarrow \frac{x}{-12-0} = \frac{-y}{36-0} = \frac{z}{-72+84}$$

$$\Rightarrow \frac{x}{-12} = \frac{y}{-36} = \frac{z}{12}$$

$$\Rightarrow \frac{x}{-1} = \frac{y}{-3} = \frac{z}{+1}$$

$$\Rightarrow X_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

When $\lambda = 9$ $\quad (A - 9I)x = 0$

$$\Rightarrow \left\{\begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 12 & -7 & 0 \\ 21 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Ans:- The Characteristic eq^n:-

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - (1+2+3)\lambda^2 + \left\{\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}\right\}\lambda - \left(1\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}\right) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + (4+5+2)\lambda - (4+2) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

When $\lambda = 1$, $1 - 6 + 11 - 6 = 0$

$\therefore \lambda = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = 2, 3.$$

$\therefore$ Eigen values are $1, 2, 3$.

When $\lambda = 1$ $\quad (A - I)x = 0$

$$\left\{\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + 0y - z &= 0 \\ x + y + z &= 0 \\ 2x + 2y + 2z &= 0 \end{aligned}\right\}$$

$$\frac{x}{0-1} = \frac{-y}{0-1} = \frac{z}{0-0}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} \qquad X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

When $\lambda = 2$, $(A-2I)X=0$

$$\Rightarrow \left\{\begin{bmatrix}1&0&-1\\1&2&1\\2&2&3\end{bmatrix}-\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\right\}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-1&0&-1\\1&0&1\\2&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned}-x+0y-z&=0\\x+0y+z&=0\\2x+2y+z&=0\end{aligned}\right\}$$

$$\Rightarrow \frac{x}{0-2}=\frac{-y}{1-2}=\frac{z}{2-0}$$

$$\Rightarrow \frac{x}{-2}=\frac{-y}{-1}=\frac{z}{2}$$

$$\Rightarrow \frac{x}{-2}=\frac{y}{1}=\frac{z}{2}$$

$$\therefore X_2=\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

When $\lambda = 3$, $(A-3I)X=0$

$$\left\{\begin{bmatrix}1&0&-1\\1&2&1\\2&2&3\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right\}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-2&0&-1\\1&-1&1\\2&2&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned}-2x+0y-z&=0\\x-y+z&=0\end{aligned}\right\}$$
$$2x+2y-z=0$$

$$\frac{x}{0-1}=\frac{-y}{-2+1}=\frac{z}{2-0}$$

$$\Rightarrow \frac{x}{-1}=\frac{-y}{-1}=\frac{z}{2}$$

$$\Rightarrow \frac{x}{-1}=\frac{y}{1}=\frac{z}{2}$$

$$X_3=\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$\therefore$ Model matrix

$$P=\begin{bmatrix}1&-2&-1\\-1&1&1\\0&2&2\end{bmatrix}$$

$$P_{11}=(-1)^2\begin{vmatrix}1&1\\2&2\end{vmatrix}=0$$

$$P_{12}=(-1)^3\begin{vmatrix}-1&1\\0&2\end{vmatrix}=+2$$

$$P_{13}=(-1)^4\begin{vmatrix}-1&1\\0&2\end{vmatrix}=-2$$

$-4+2$

$$P_{21}=(-1)^3\begin{vmatrix}-2&-1\\2&2\end{vmatrix}=-(-2)=2$$

$$P_{22}=(-1)^4\begin{vmatrix}1&-1\\0&2\end{vmatrix}=2$$

$$P_{23}=(-1)^5\begin{vmatrix}1&-2\\0&2\end{vmatrix}=-2$$

$$P_{31}=(-1)^4\begin{vmatrix}-2&-1\\1&1\end{vmatrix}=-1$$

$$P_{32}=(-1)^5\begin{vmatrix}1&-1\\-1&1\end{vmatrix}=-(2)\times 2\times 0$$

$$P_{33}=(-1)^6\begin{vmatrix}1&-2\\-1&1\end{vmatrix}=\cancel{3}\;-1$$

$$\therefore \text{Cofactor matrix}=\begin{bmatrix}0&-2&2\\2&2&-2\\-1&-2&3\end{bmatrix}$$

$$\text{adj } P=\begin{bmatrix}0&2&-1\\-2&2&-2\\2&-2&3\cancel{2}\end{bmatrix}$$

$$|P|=1\begin{vmatrix}1&1\\2&2\end{vmatrix}+1\begin{vmatrix}-2&-1\\2&2\end{vmatrix}$$

$-4+2$

$$=0+\cancel{(-2)}\cancel{-2}\;-2$$

Que 3:- Diagonalize the given matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Ans:- The characteristic eqⁿ:-

$$\lambda^3 - (-2+1+0)\lambda^2 + \left\{ \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} \right\}\lambda - \left(-2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \right) = 0$$

$\Rightarrow \lambda^3 + \lambda^2 + (-12-3-6)\lambda - (-2\times -12 - 2\times -6 - 3\times -3) = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - (24+12+9) = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

when $\lambda = 1$, $1+1-21-45 \neq 0$

when $\lambda = 5$, $125+25-105-45 = 0$

$\therefore \lambda = 5$ is a root.

$$\begin{array}{r|rrrr} 5 & 1 & 1 & -21 & -45 \\ & 0 & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$\Rightarrow \lambda^2 + 6\lambda + 9 = 0$

$\Rightarrow (\lambda+3)(\lambda+3) = 0$

$\Rightarrow \lambda = -3, -3.$

$\therefore$ The eigen values are

$5, -3, -3$

when $\lambda = 5$ $(A-5I)X=0$

$$\Rightarrow \left\{\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix} - \begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}\right\}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned}-7x+2y-3z&=0\\ 2x-4y-6z&=0\end{aligned}\right\}$$

$$-x-2y-5z=0$$

$$\Rightarrow \frac{x}{-12-12} = \frac{-y}{42+6} = \frac{z}{28-4}$$

$$\Rightarrow \frac{x}{-24} = \frac{-y}{48} = \frac{z}{24}$$

$$\Rightarrow \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$X_1 = \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

when $\lambda = -3$ $(A+3I)X=0$

$$\left\{\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix} + \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}\right\}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x+2y-3z=0.$$
$$2x+4y-6z=0.$$
$$-x-2y+3z=0.$$

$$\Rightarrow x+2y-3z=0$$
$$x+2y-3z=0$$
$$x+2y-3z=0.$$

$\therefore$ let $z=t_1$ and $y=t_2$

$$\Rightarrow x+2t_2-3t_1=0$$
$$\Rightarrow x = 3t_1-2t_2.$$

$$\therefore X = \begin{bmatrix}3t_1-2t_2\\ t_2\\ t_1\end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix}3t_1\\0\\t_1\end{bmatrix} + \begin{bmatrix}-2t_2\\t_2\\0\end{bmatrix}$$

$$\Rightarrow X = t_1\begin{bmatrix}3\\0\\1\end{bmatrix} + t_2\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix}3\\0\\1\end{bmatrix}, \quad X_3 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$$

$\therefore$ The model matrix $P = [X_1\ X_2\ X_3]$

$$\Rightarrow P = \begin{bmatrix}1 & 3 & -2\\ 2 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}$$

$$P_{11} = (-1)^2\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix} = -1$$

$$P_{12} = (-1)^3\begin{vmatrix}2 & 1\\ -1 & 0\end{vmatrix} = -1$$

$$P_{13} = (-1)^4\begin{vmatrix}2 & 0\\ -1 & 1\end{vmatrix} = 2$$

$$P_{21} = (-1)^3\begin{vmatrix}3 & -2\\ 1 & 0\end{vmatrix} = -2$$

$$P_{22} = (-1)^4\begin{vmatrix}1 & -2\\ -1 & 0\end{vmatrix} = -2$$

$$P_{23} = (-1)^5\begin{vmatrix}1 & 3\\ -1 & 1\end{vmatrix} = -4$$

$$P_{31} = (-1)^4\begin{vmatrix}3 & -2\\ 0 & 1\end{vmatrix} = 3$$

$$P_{32} = (-1)^5 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -5$$

$$P_{33} = (-1)^6 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6.$$

$$\therefore \text{Adj } P = \begin{bmatrix} -1 & -2 & 3 \\ -1 & -2 & -5 \\ 2 & -4 & -6 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= (-1 \times 3) - (1 \times 5)$$

$$= -3 - 5 = \underline{-8}$$

$$\therefore P^{-1} = \frac{1}{-8} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -2 & -5 \\ 2 & -4 & -6 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}$$

$$\Rightarrow P^{-1} A P$$

$$= \frac{1}{8} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -9 & 6 \\ 10 & 0 & -3 \\ -5 & -3 & 0 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 40 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = D$$

Que 4:- Diagonalize the given matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Ans:- The characteristic eqn:-

$$\lambda^3 - (2+2+2)\lambda^2 + \left\{ \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \right\} \lambda - \left\{ 2 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \right\} = 0.$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + (4+3+4) - (2 \times 4 + 1 \times -2)$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

when $\lambda = 1$, $1 - 6 + 11 - 6 = 0$

$\therefore \lambda = 1$ is a root.

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0.$$

$$\therefore \lambda = 2, 3.$$

ie, $\lambda = 1, 2, 3$.

when $\lambda = 1$, $(A - I)x = 0$

$$\left\{ \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} x + 0y + z = 0. \\ 0x + y + 0z = 0 \\ x + 0y + z = 0 \end{array} \right\}$$

$$\frac{x}{1} = \frac{-y}{0} = \frac{z}{-1}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

When $\lambda = 2$, $(A-2I)X = 0$

$$\left\{\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix} - \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\right\}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$0x + 0y - z = 0$ ✓

$0x + 0y + 0z = 0$

$x + 0y + 0z = 0$ ✓

$$\Rightarrow \frac{x}{0-0} = \frac{-y}{0+1} = \frac{z}{0-0}$$

$$\Rightarrow \frac{x}{0} = \frac{y}{-1} = \frac{z}{0}$$

$$\Rightarrow \frac{x}{0} = \frac{y}{1} = \frac{z}{0}$$

$$\therefore X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

When $\lambda = 3$, $(A-3I)X = 0$.

$$\left\{\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix} - \begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right\}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-1&0&1\\0&-1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow -x + 0y + z = 0$

$\left.\begin{aligned}0x - y + 0z &= 0\\ x + 0y - z &= 0\end{aligned}\right\}$

$$\frac{x}{1-0} = \frac{-y}{0-0} = \frac{z}{0+1}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\therefore P = \begin{bmatrix}1&0&1\\0&1&0\\-1&0&1\end{bmatrix}$$

$P_{11} = (-1)^2\begin{vmatrix}1&0\\0&1\end{vmatrix} = 1$

$P_{12} = (-1)^3\begin{vmatrix}0&0\\-1&1\end{vmatrix} = 0$.

$P_{13} = (-1)^4\begin{vmatrix}0&1\\-1&0\end{vmatrix} = 1$

$P_{21} = (-1)^3\begin{vmatrix}0&1\\0&1\end{vmatrix} = 0$

$P_{22} = (-1)^4\begin{vmatrix}1&1\\-1&1\end{vmatrix} = 2$.

$P_{23} = (-1)^5\begin{vmatrix}1&0\\-1&0\end{vmatrix} = 0$.

$P_{31} = (-1)^4\begin{vmatrix}0&1\\1&0\end{vmatrix} = -1$

$P_{32} = (-1)^5\begin{vmatrix}1&1\\0&0\end{vmatrix} = 0$

$P_{33} = (-1)^6\begin{vmatrix}1&0\\0&1\end{vmatrix} = 1$

$$\therefore \text{Adj } P = \begin{bmatrix}1&0&-1\\0&2&0\\1&0&1\end{bmatrix}$$

$$|P| = 1\begin{vmatrix}1&1\\-1&1\end{vmatrix} = 2.$$

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix}1&0&-1\\0&2&0\\1&0&1\end{bmatrix}\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}\begin{bmatrix}1&0\\0&1\\-1&0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1&0&-1\\0&2&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&3\\0&2&0\\-1&0&3\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Excercise.

I) Diagonalize the following matrices:-

(i) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Eigen values = 8, 2, 2.

Eigen vectors = $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Eigen values = 1, 1, 4

Eigen vectors = $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Eigen values = 2, 2, 5

vectors = $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$