Module 1

Arguments and Venn Diagrams

EXAMPLE 1.3 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:

S_1: All my tin objects are saucepans.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S : Your presents to me are not made of tin.

The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.

By S_1 the tin objects are contained in the set of saucepans, and by S_3 the set of saucepans and the set of useful things are disjoint. Furthermore, by S_2 the set of "your presents" is a subset of the set of useful things. Accordingly, we can draw the Venn diagram in Fig. 1-2.

The conclusion is clearly valid by the Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

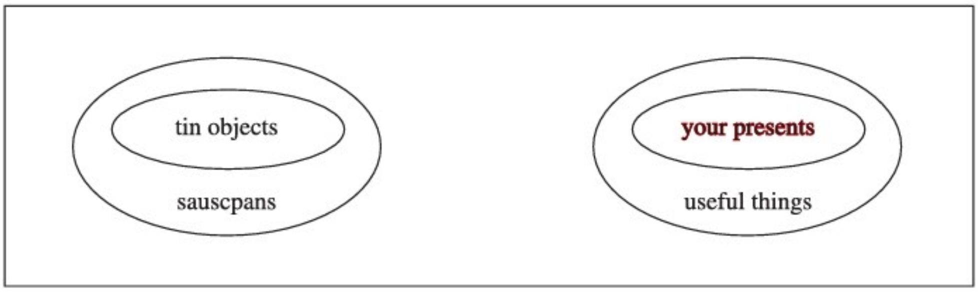


Fig. 1-2

EXAMPLE 1.5 Suppose $\mathbf{U} = \mathbf{N} = \{1, 2, 3, \ldots\}$ is the universal set. Let

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{2, 3, 8, 9\}, \quad E = \{2, 4, 6, \ldots\}$$

(Here E is the set of even integers.) Then:

$$A^{\mathrm{C}} = \{5, 6, 7, \ldots\}, \quad B^{\mathrm{C}} = \{1, 2, 8, 9, 10, \ldots\}, \quad E^{\mathrm{C}} = \{1, 3, 5, 7, \ldots\}$$

That is, E^{C} is the set of odd positive integers. Also:

$$\begin{array}{llll} A\backslash B = \{1, 2\}, & A\backslash C = \{1, 4\}, & B\backslash C = \{4, 5, 6, 7\}, & A\backslash E = \{1, 3\}, \\ B\backslash A = \{5, 6, 7\}, & C\backslash A = \{8, 9\}, & C\backslash B = \{2, 8, 9\}, & E\backslash A = \{6, 8, 10, 12, \ldots\}. \end{array}$$

Furthermore:

$$\begin{array}{ll} A \oplus B = (A\backslash B) \cup (B\backslash A) = \{1, 2, 5, 6, 7\}, & B \oplus C = \{2, 4, 5, 6, 7, 8, 9\}, \\ A \oplus C = (A\backslash C) \cup (B\backslash C) = \{1, 4, 8, 9\}, & A \oplus E = \{1, 3, 6, 8, 10, \ldots\}. \end{array}$$

EXAMPLE 1.7

(a) The set A of the letters of the English alphabet and the set D of the days of the week are finite sets. Specifically, A has 26 elements and D has 7 elements.

(b) Let E be the set of even positive integers, and let $\mathbf{I}$ be the *unit interval*, that is,

$$E = \{2, 4, 6, \ldots\} \quad \text{and} \quad \mathbf{I} = [0, 1] = \{x \mid 0 \le x \le 1\}$$

Then both E and $\mathbf{I}$ are infinite.

A set $\boldsymbol{S}$ is *countable* if $\boldsymbol{S}$ is finite or if the elements of $\boldsymbol{S}$ can be arranged as a sequence, in which case $\boldsymbol{S}$ is said to be *countably infinite*; otherwise $\boldsymbol{S}$ is said to be *uncountable*. The above set E of even integers is countably infinite, whereas one can prove that the unit interval $\mathbf{I} = [0, 1]$ is uncountable.

Inclusion–Exclusion Principle

There is a formula for $n(A \cup B)$ even when they are not disjoint, called the Inclusion–Exclusion Principle. Namely:

Theorem (Inclusion–Exclusion Principle) 1.9: Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to obtain a similar formula for three sets:

Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

EXAMPLE 1.8 Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students: (a) only on list A, (b) only on list B, (c) on list A or B (or both), (d) on exactly one list.

(a) List A has 30 names and 20 are on list B; hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

In other words, we combine the two lists and then cross out the 20 names which appear twice.

(d) By (a) and (b), $10 + 15 = 25$ names are only on one list; that is, $n(A \oplus B) = 25$.

Power Sets

For a given set S, we may speak of the class of all subsets of S. This class is called the *power set* of S, and will be denoted by $P(S)$. If S is finite, then so is $P(S)$. In fact, the number of elements in $P(S)$ is 2 raised to the power $n(S)$. That is,

$$n(P(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.)

EXAMPLE 1.10 Suppose $S = \{1, 2, 3\}$. Then

$$P(S) = [\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S]$$

Note that the empty set $\emptyset$ belongs to $P(S)$ since $\emptyset$ is a subset of S. Similarly, S belongs to $P(S)$. As expected from the above remark, $P(S)$ has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A *partition* of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.
(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_j \neq A_k \quad \text{then} \quad A_j \cap A_k = \emptyset$$

The subsets in a partition are called *cells*. Figure 1-6 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, A_5.

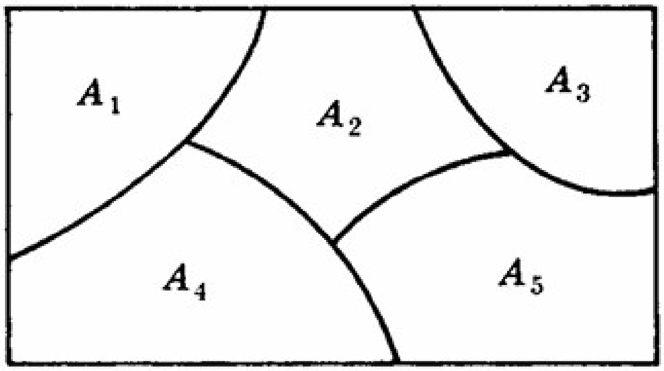


Fig. 1-6

EXAMPLE 1.11 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$

(ii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$

(iii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. On the other hand, (iii) is a partition of S.

1.8 MATHEMATICAL INDUCTION

An essential property of the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers follows:

Principle of Mathematical Induction I: Let P be a proposition defined on the positive integers $\mathbf{N}$; that is, $P(n)$ is either true or false for each $n \in \mathbf{N}$. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(k+1)$ is true whenever $P(k)$ is true.

Then P is true for every positive integer $n \in \mathbf{N}$.

EXAMPLE 1.13 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(The kth odd number is $2k - 1$, and the next odd number is $2k + 1$.) Observe that $P(n)$ is true for $n = 1$; namely,

$$P(1) = 1^2$$

Assuming $P(k)$ is true, we add $2k + 1$ to both sides of $P(k)$, obtaining

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) - k^2 + (2k + 1) = (k + 1)^2$$

which is $P(k + 1)$. In other words, $P(k + 1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.

Principle of Mathematical Induction II: Let P be a proposition defined on the positive integers **N** such that:
(i) $P(1)$ is true.
(ii) $P(k)$ is true whenever $P(j)$ is true for all $1 \le j < k$.
Then P is true for every positive integer $n \in \mathbf{N}$.

Solved Problems

SETS AND SUBSETS

1.1 Which of these sets are equal: $\{x, y, z\}$, $\{z, y, z, x\}$, $\{y, x, y, z\}$, $\{y, z, x, y\}$?

They are all equal. Order and repetition do not change a set.

1.2 List the elements of each set where $\mathbf{N} = \{1, 2, 3, \ldots\}$.

(*a*) $A = \{x \in \mathbf{N} \mid 3 < x < 9\}$

(*b*) $B = \{x \in \mathbf{N} \mid x \text{ is even}, x < 11\}$

(*c*) $C = \{x \in \mathbf{N} \mid 4 + x = 3\}$

(*a*) A consists of the positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.

(*b*) B consists of the even positive integers less than 11; hence $B = \{2, 4, 6, 8, 10\}$.

(*c*) No positive integer satisfies $4 + x = 3$; hence $C = \emptyset$, the empty set.

1.3 Let $A = \{2, 3, 4, 5\}$.

(a) Show that A is not a subset of $B = \{x \in \mathbf{N} \mid x \text{ is even}\}$.

(b) Show that A is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

(a) It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

(b) Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

1.5 Consider the sets in the preceding Problem 1.4. Find:

(a) $A^{C}, B^{C}, D^{C}, E^{C}$; ($b$) $A \backslash B, B \backslash A, D \backslash E$; ($c$) $A \oplus B,\ C \oplus D,\ E \oplus F$.

Recall that:

(1) The complements X^{C} consists of those elements in $\mathbf{U}$ which do not belong to X.

(2) The difference $X \backslash Y$ consists of the elements in X which do not belong to Y.

(3) The symmetric difference $X \oplus Y$ consists of the elements in X or in Y but not in both.

Therefore:

(a) $A^{C} = \{6, 7, 8, 9\}$; $B^{C} = \{1, 2, 3, 8, 9\}$; $D^{C} = \{2, 4, 6, 8\} = E$; $E^{C} = \{1, 3, 5, 7, 9\} = D$.

(b) $A \backslash B = \{1, 2, 3\}$; $B \backslash A = \{6, 7\}$; $D \backslash E = \{1, 3, 5, 7, 9\} = D$; $F \backslash D = \emptyset$.

(c) $A \oplus B = \{1, 2, 3, 6, 7\}$; $C \oplus D = \{1, 3, 6, 8\}$; $E \oplus F = \{2, 4, 6, 8, 1, 5, 9\} = E \cup F$.

1.6 Show that we can have: (a) $A \cap B = A \cap C$ without $B = C$; (b) $A \cup B = A \cup C$ without $B = C$.

(a) Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{2, 4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$; but $B \neq C$.

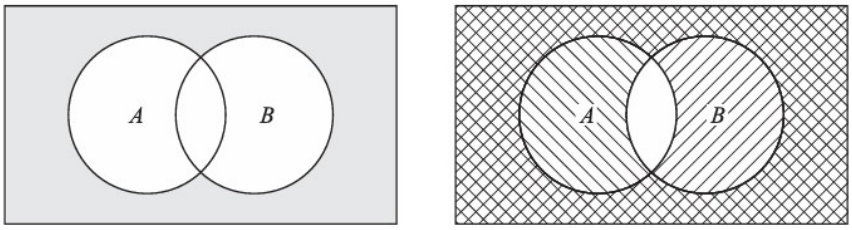
(b) Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$ and $A \cup C = \{1, 2, 3\}$ but $B \neq C$.

1.7 Prove: $B \backslash A = B \cap A^{C}$. Thus, the set operation of difference can be written in terms of the operations of intersection and complement.

$$B \backslash A = \{x \mid x \in B,\ x \notin A\} = \{x \mid x \in B,\ x \in A^{C}\} = B \cap A^{C}.$$

1.9 Illustrate DeMorgan's Law $(A \cup B)^C = A^C \cap B^C$ using Venn diagrams.

Shade the area outside $A \cup B$ in a Venn diagram of sets A and B. This is shown in Fig. 1-7(a); hence the shaded area represents $(A \cup B)^C$. Now shade the area outside A in a Venn diagram of A and B with strokes in one direction (////), and then shade the area outside B with strokes in another direction (\\\\\\\\). This is shown in Fig. 1-7(b); hence the cross-hatched area (area where both lines are present) represents $A^C \cap B^C$. Both $(A \cup B)^C$ and $A^C \cap B^C$ are represented by the same area; thus the Venn diagram indicates $(A \cup B)^C = A^C \cap B^C$. (We emphasize that a Venn diagram is not a formal proof, but it can indicate relationships between sets.)



(a) (b)

1.13 Determine the validity of the following argument:

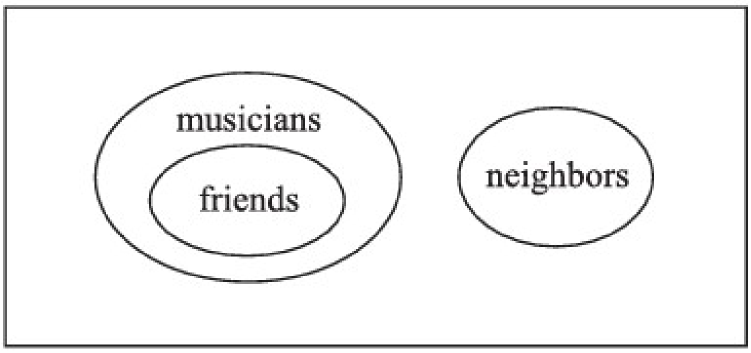
S_1: All my friends are musicians.

S_2: John is my friend.

S_3: None of my neighbors are musicians.

S : John is not my neighbor.

The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-8(a). By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.



(a)

Fig. 1-8

1.14 Each student in Liberal Arts at some college has a mathematics requirement A and a science requirement B. A poll of 140 sophomore students shows that:

60 completed A, 45 completed B, 20 completed both A and B.

Use a Venn diagram to find the number of students who have completed:

(a) At least one of A and B; (b) exactly one of A or B; (c) neither A nor B.

Translating the above data into set notation yields:

$n(A) = 60,\ n(B) = 45,\ n(A \cap B) = 20,\ n(\mathbf{U}) = 140$

Draw a Venn diagram of sets A and B as in Fig. 1-1(c). Then, as in Fig. 1-8(b), assign numbers to the four regions as follows:

20 completed both A and B, so $n(A \cap B) = 20$.

$60 - 20 = 40$ completed A but not B, so $n(A \backslash B) = 40$.

$45 - 20 = 25$ completed B but not A, so $n(B \backslash A) = 25$.

$140 - 20 - 40 - 25 = 55$ completed neither A nor B.

By the Venn diagram:

(a) $20 + 40 + 25 = 85$ completed A or B. Alternately, by the Inclusion–Exclusion Principle:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 60 + 45 - 20 = 85$$

(b) $40 + 25 = 65$ completed exactly one requirement. That is, $n(A \oplus B) = 65$.

(c) 55 completed neither requirement, i.e. $n(A^{\mathrm{C}} \cap B^{\mathrm{C}}) = n[(A \cup B)^{\mathrm{C}}] = 140 - 85 = 55$.

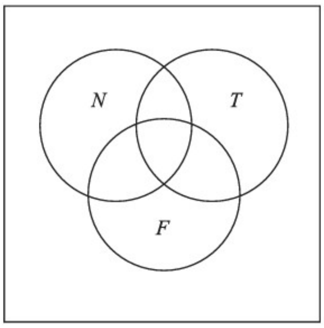
1.15 In a survey of 120 people, it was found that:

65 read *Newsweek* magazine, 20 read both *Newsweek* and *Time*,
45 read *Time*, 25 read both *Newsweek* and *Fortune*,
42 read *Fortune*, 15 read both *Time* and *Fortune*,
8 read all three magazines.

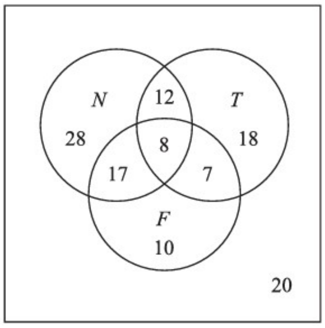
(a) Find the number of people who read at least one of the three magazines.

(b) Fill in the correct number of people in each of the eight regions of the Venn diagram in Fig. 1-9(a) where N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune*, respectively.

(c) Find the number of people who read exactly one magazine.



(a)



(b)

Fig. 1-9

(a) We want to find $n(N \cup T \cup F)$. By Corollary 1.10 (Inclusion–Exclusion Principle),

$$\begin{aligned} n(N \cup T \cup F) &= n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

(b) The required Venn diagram in Fig. 1-9(b) is obtained as follows:

8 read all three magazines,

$20 - 8 = 12$ read *Newsweek* and *Time* but not all three magazines,

$25 - 8 = 17$ read *Newsweek* and *Fortune* but not all three magazines,

$15 - 8 = 7$ read *Time* and *Fortune* but not all three magazines,

$65 - 12 - 8 - 17 = 28$ read only *Newsweek*,

$45 - 12 - 8 - 7 = 18$ read only *Time*,

$42 - 17 - 8 - 7 = 10$ read only *Fortune*,

$120 - 100 = 20$ read no magazine at all.

(c) $28 + 18 + 10 = 56$ read exactly one of the magazines.

1.18 Determine the power set $P(A)$ of $A = \{a, b, c, d\}$.

The elements of $P(A)$ are the subsets of A. Hence

$$P(A) = [A, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset]$$

As expected, $P(A)$ has $2^4 = 16$ elements.

1.19 Let $S = \{a, b, c, d, e, f, g\}$. Determine which of the following are partitions of S:

(a) $P_1 = [\{a, c, e\}, \{b\}, \{d, g\}]$, (c) $P_3 = [\{a, b, e, g\}, \{c\}, \{d, f\}]$,

(b) $P_2 = [\{a, e, g\}, \{c, d\}, \{b, e, f\}]$, (d) $P_4 = [\{a, b, c, d, e, f, g\}]$.

(a) P_1 is not a partition of S since $f \in S$ does not belong to any of the cells.

(b) P_2 is not a partition of S since $e \in S$ belongs to two of the cells.

(c) P_3 is a partition of S since each element in S belongs to exactly one cell.

(d) P_4 is a partition of S into one cell, S itself.

1.20 Find all partitions of $S = \{a, b, c, d\}$.

Note first that each partition of S contains either 1, 2, 3, or 4 distinct cells. The partitions are as follows:

(1) $[\{a, b, c, d\}]$

(2) $[\{a\}, \{b, c, d\}]$, $[\{b\}, \{a, c, d\}]$, $[\{c\}, \{a, b, d\}]$, $[\{d\}, \{a, b, c\}]$,
$[\{a, b\}, \{c, d\}]$, $[\{a, c\}, \{b, d\}]$, $[\{a, d\}, \{b, c\}]$

(3) $[\{a\}, \{b\}, \{c, d\}]$, $[\{a\}, \{c\}, \{b,d\}]$, $[\{a\}, \{d\}, \{b, c\}]$,
$[\{b\}, \{c\}, \{a, d\}]$, $[\{b\}, \{d\}, \{a, c\}]$, $[\{c\}, \{d\}, \{a, b\}]$

(4) $[\{a\}, \{b\}, \{c\}, \{d\}]$

There are 15 different partitions of S.

1.21 Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and, for each $n \in \mathbf{N}$, Let $A_n = \{n, 2n, 3n, \ldots\}$. Find:

(a) $A_3 \cap A_5$; (b) $A_4 \cap A_5$; (c) $\bigcup_{i \in Q} A_i$ where $Q = \{2, 3, 5, 7, 11, \ldots\}$ is the set of prime numbers.

(a) Those numbers which are multiples of both 3 and 5 are the multiples of 15; hence $A_3 \cap A_5 = A_{15}$.

(b) The multiples of 12 and no other numbers belong to both A_4 and A_6, hence $A_4 \cap A_6 = A_{12}$.

(c) Every positive integer except 1 is a multiple of at least one prime number; hence

$$\bigcup_{i \in Q} A_i = \{2, 3, 4, \ldots\} = \mathbf{N} \backslash \{1\}$$

MATHEMATICAL INDUCTION

1.24 Prove the proposition $P(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n) = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

The proposition holds for $n = 1$ since:

$$P(1) : 1 = \tfrac{1}{2}(1)(1+1)$$

Assuming $P(k)$ is true, we add $k+1$ to both sides of $P(k)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \tfrac{1}{2}k(k+1) + (k+1) \\ &= \tfrac{1}{2}[k(k+1) + 2(k+1)] \\ &= \tfrac{1}{2}[(k+1)(k+2)] \end{aligned}$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the Principle of Induction, P is true for all n.

Relations

EXAMPLE 2.2 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a),\ (1, b),\ (1, c),\ (2, a),\ (2, b),\ (2, c)\}$$
$$B \times A = \{(a, 1),\ (b, 1),\ (c, 1),\ (a, 2),\ (b, 2),\ (c, 2)\}$$

Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

EXAMPLE 2.3

(a) $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry, 1Rz, 3Ry, \quad \text{but} \quad 1\not{R}x, 2\not{R}x, 2\not{R}y, 2\not{R}z, 3\not{R}x, 3\not{R}z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(b) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of set A and B, either $A \subseteq B$ or $A \nsubseteq B$.

(c) A familiar relation on the set **Z** of integers is "m divides n." A common notation for this relation is to write $m \mid n$ when m divides n. Thus $6 \mid 30$ but $7 \nmid 25$.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

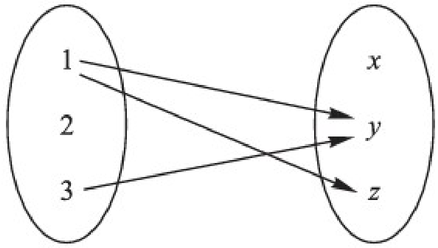
For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Then the inverse of

$$R = \{(1, y), (1, z), (3, y)\} \quad \text{is} \quad R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$$

PICTURE REPRESENTATION OF RELATION

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Fig. 2-3

2.2. Find x and y given $(2x, x + y) = (6, 2)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations

$$2x = 6 \quad \text{and} \quad x + y = 2$$

from which we derive the answers $x = 3$ and $y = -1$.

2.3. Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

There are $3(2) = 6$ elements in $A \times B$, and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus there are $m = 64$ relations from A to B.

2.4. Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(*a*) Determine the matrix of the relation.

(*b*) Draw the arrow diagram of R.

(*c*) Find the inverse relation R^{-1} of R.

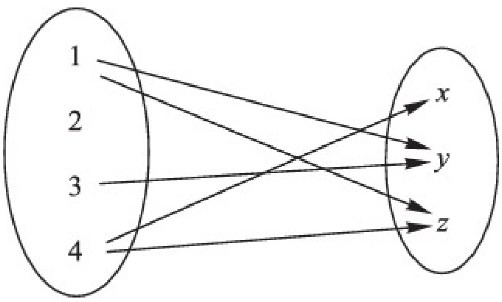
(*d*) Determine the domain and range of R.

(*a*) See Fig. 2-6(*a*) Observe that the rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that the entry in the matrix corresponding to $a \in A$ and $b \in B$ is 1 if a is related to b and 0 otherwise.

(*b*) See Fig. 2.6(*b*) Observe that there is an arrow from $a \in A$ to $b \in B$ iff a is related to b, i.e., iff $(a, b) \in R$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

(*a*)



(*b*)

Fig. 2-6

(*c*) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

Observe that by reversing the arrows in Fig. 2.6(*b*), we obtain the arrow diagram of R^{-1}.

(*d*) The domain of R, $\text{Dom}(R)$, consists of the first elements of the ordered pairs of R, and the range of R, $\text{Ran}(R)$, consists of the second elements. Thus,

$$\text{Dom}(R) = \{1, 3, 4\} \quad \text{and} \quad \text{Ran}(R) = \{x, y, z\}$$

2.5. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C, respectively.

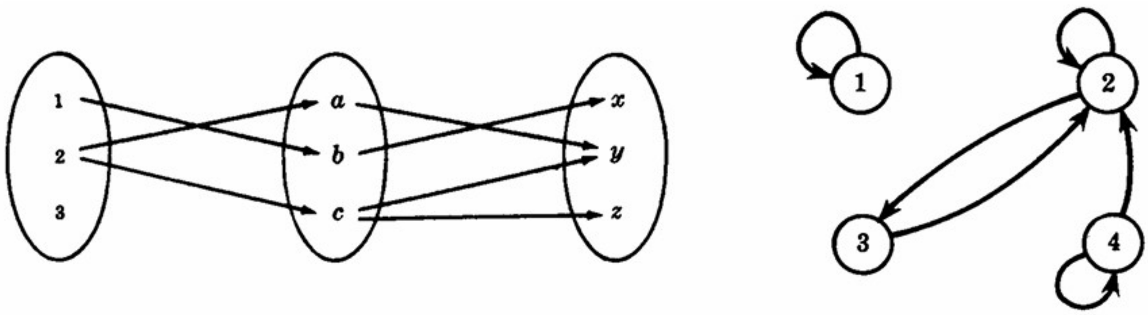
$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $R \circ S$.

(b) Find the matrices M_R, M_S, and $M_{R\circ S}$ of the respective relations R, S, and $R \circ S$, and compare $M_{R\circ S}$ to the product $M_R M_S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2-7(a). Observe that 1 in A is "connected" to x in C by the path $1 \to b \to x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$.
We have

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$



(a) (b)

Fig. 2-7

(b) The matrices of M_R, M_S, and $M_{R\circ S}$ follow:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \quad M_{R\circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe that $M_{R\circ S}$ and $M_R M_S$ have the same zero entries.

TYPES OF RELATIONS AND CLOSURE PROPERTIES

2.9. Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}, \qquad \emptyset = \text{empty relation}$$
$$S = \{(1, 1)(1, 2), (2, 1)(2, 2), (3, 3)\}, \qquad A \times A = \text{universal relation}$$
$$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$$

Determine whether or not each of the above relations on A is: (*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

(*a*) R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\emptyset$ is not reflexive. S and $A \times A$ are reflexive.

(*b*) R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\emptyset$, and $A \times A$ are symmetric.

(*c*) T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four relations are transitive.

(*d*) S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.10. Give an example of a relation R on $A = \{1, 2, 3\}$ such that:

(*a*) R is both symmetric and antisymmetric.

(*b*) R is neither symmetric nor antisymmetric.

(*c*) R is transitive but $R \cup R^{-1}$ is not transitive.

There are several such examples. One possible set of examples follows:

(*a*) $R = \{(1, 1), (2, 2)\}$;   (*b*) $R = \{(1, 2), (2, 3)\}$;   (*c*) $R = \{(1, 2)\}$.

2.13. Consider the relation $R = \{(a, a), (a, b), (b, c), (c, c)\}$ on the set $A = \{a, b, c\}$. Find: (a) reflexive(R); (b) symmetric(R); (c) transitive(R).

(a) The reflexive closure on R is obtained by adding all diagonal pairs of $A \times A$ to R which are not currently in R. Hence,

$$\text{reflexive}(R) = R \cup \{(b, b)\} = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$$

(b) The symmetric closure on R is obtained by adding all the pairs in R^{-1} to R which are not currently in R. Hence,

$$\text{symmetric}(R) = R \cup \{(b, a), (c, b)\} = \{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$$

(c) The transitive closure on R, since A has three elements, is obtained by taking the union of R with $R^2 = R \circ R$ and $R^3 = R \circ R \circ R$. Note that

$$R^2 = R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$
$$R^3 = R \circ R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

Hence

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

EQUIVALENCE RELATIONS AND PARTITIONS

2.14. Consider the **Z** of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod{m}$$

if $x - y$ is divisible by m. Show that this defines an equivalence relation on **Z**.

We must show that the relation is reflexive, symmetric, and transitive.

(i) For any x in Z we have $x \equiv x \pmod{m}$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

(ii) Suppose $x \equiv y \pmod{m}$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod{m}$. Thus the relation is symmetric.

(iii) Now suppose $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod{m}$. Thus the relation is transitive.

Accordingly, the relation of congruence modulo m on **Z** is an equivalence relation.

2.15. Let A be a set of nonzero integers and let $\approx$ be the relation on $A \times A$ defined by

$$(a, b) \approx (c, d) \quad \text{whenever} \quad ad = bc$$

Prove that $\approx$ is an equivalence relation.

We must show that $\approx$ is reflexive, symmetric, and transitive.

(i) *Reflexivity*: We have $(a, b) \approx (a, b)$ since $ab = ba$. Hence $\approx$ is reflexive.

(ii) *Symmetry*: Suppose $(a, b) \approx (c, d)$. Then $ad = bc$. Accordingly, $cb = da$ and hence $(c, d) = (a, b)$. Thus, $\approx$ is symmetric.

(iii) *Transitivity*: Suppose $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying corresponding terms of the equations gives $(ad)(cf) = (bc)(de)$. Canceling $c \neq 0$ and $d \neq 0$ from both sides of the equation yields $af = be$, and hence $(a, b) \approx (e, f)$. Thus $\approx$ is transitive. Accordingly, $\approx$ is an equivalence relation.

2.16. Let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$:

$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$

Find the partition of A induced by R, i.e., find the equivalence classes of R.

Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, 3, and 6, hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4. Thus

$$[4] = \{4\}$$

Accordingly, the following is the partition of A induced by R:

$$[\{1, 5\}, \{2, 3, 6\}, \{4\}]$$

PARTIAL ORDERINGS

2.18. Let ℓ be any collection of sets. Is the relation of set inclusion $\subseteq$ a partial order on ℓ?

Yes, since set inclusion is reflexive, antisymmetric, and transitive. That is, for any sets A, B, C in ℓ we have: (i) $A \subseteq A$; (ii) if $A \subseteq B$ and $B \subseteq A$, then $A = B$; (iii) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

2.19. Consider the set **Z** of integers. Define aRb by $b = a^r$ for some positive integer r. Show that R is a partial order on **Z**, that is, show that R is: (*a*) reflexive; (*b*) antisymmetric; (*c*) transitive.

(*a*) R is reflexive since $a = a^1$.

(*b*) Suppose aRb and bRa, say $b = a^r$ and $a = b^s$. Then $a = (a^r)^s = a^{rs}$. There are three possibilities: (i) $rs = 1$, (ii) $a = 1$, and (iii) $a = -1$. If $rs = 1$ then $r = 1$ and $s = 1$ and so $a = b$. If $a = 1$ then $b = 1^r = 1 = a$, and, similarly, if $b = 1$ then $a = 1$. Lastly, if $a = -1$ then $b = -1$ (since $b \neq 1$) and $a = b$. In all three cases, $a = b$. Thus R is antisymmetric.

(*c*) Suppose aRb and bRc say $b = a^r$ and $c = b^s$. Then $c = (a^r)^s = a^{rs}$ and, therefore, aRc. Hence R is transitive.

Accordingly, R is a partial order on **Z**.

Functions

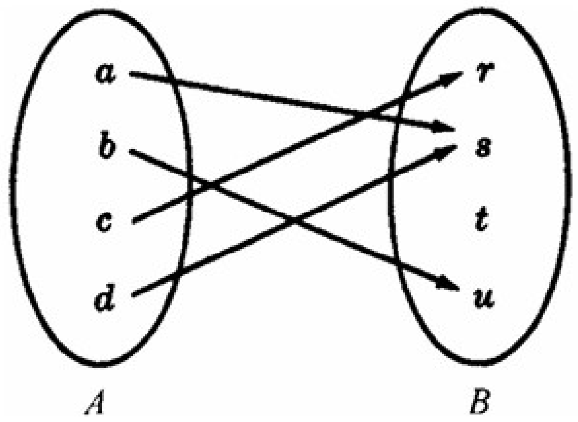


Fig. 3-1

EXAMPLE 3.1

(a) Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

(b) Figure 3-1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way. Here

$$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$$

The image of f is the set of image values, $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element under f.

(c) Let A be any set. The function from A into A which assigns to each element in A the element itself is called the *identity function* on A and it is usually denoted by 1_A, or simply 1. In other words, for every $a \in A$,

$$1_A(a) = a.$$

Composition Function

Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) \equiv g(f(a))$$

3.3 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A onto B or that f maps A onto B.

A function $f: A \to B$ is *invertible* if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function. The following theorem gives simple criteria which tells us when it is.

Theorem 3.1: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

Floor and Ceiling Functions

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically,

$\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x.

$\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$. For example,

$$\lfloor 3.14 \rfloor = 3, \quad \left\lfloor \sqrt{5} \right\rfloor = 2, \quad \lfloor -8.5 \rfloor = -9, \quad \lfloor 7 \rfloor = 7, \quad \lfloor -4 \rfloor = -4,$$

$$\lceil 3.14 \rceil = 4, \quad \left\lceil \sqrt{5} \right\rceil = 3, \quad \lceil -8.5 \rceil = -8, \quad \lceil 7 \rceil = 7, \quad \lceil -4 \rceil = -4$$

3.6 RECURSIVELY DEFINED FUNCTIONS

A function is said to be *recursively defined* if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties:

(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.

A recursive function with these two properties is said to be *well-defined*.

The following examples should help clarify these ideas.

Factorial Function

The product of the positive integers from 1 to n, inclusive, is called "n factorial" and is usually denoted by $n!$. That is,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

Solved Problems

FUNCTIONS

3.1. Let $X = \{1, 2, 3, 4\}$. Determine whether each relation on X is a function from X into X.

(*a*) $f = \{(2, 3), (1, 4), (2, 1), (3.2), (4, 4)\}$

(*b*) $g = \{(3, 1), (4, 2), (1, 1)\}$

(*c*) $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$

Recall that a subset f of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.

(*a*) No. Two different ordered pairs (2, 3) and (2, 1) in f have the same number 2 as their first coordinate.

(*b*) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.

(*c*) Yes. Although $2 \in X$ appears as the first coordinate in two ordered pairs in h, these two ordered pairs are equal.

3.3. Let $A = \{a, b, c\}$, $B = \{x, y, z\}$, $C = \{r, s, t\}$. Let $f: A \to B$ and $g: B \to C$ be defined by:

$$f = \{(a, y)(b, x), (c, y)\} \quad \text{and} \quad g = \{(x, s), (y, t), (z, r)\}.$$

Find: (*a*) composition function $g \circ f: A \to C$; (*b*) $\mathrm{Im}(f)$, $\mathrm{Im}(g)$, $\mathrm{Im}(g \circ f)$.

(*a*) Use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

That is $g \circ f = \{(a, t), (b, s), (c, t)\}$.

(*b*) Find the image points (or second coordinates):

$$\mathrm{Im}(f) = \{x, y\}, \quad \mathrm{Im}(g) = \{r, s, t\}, \quad \mathrm{Im}(g \circ f) = \{s, t\}$$

1.25 Prove the following proposition (for $n \geq 0$):

$$P(n): 1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

$P(0)$ is true since $1 = 2^1 - 1$. Assuming $P(k)$ is true, we add 2^{k+1} to both sides of $P(k)$, obtaining

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the principle of induction, $P(n)$ is true for all n.