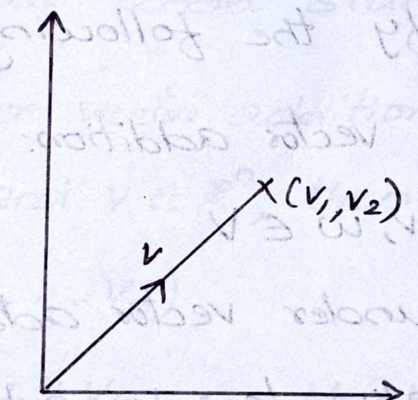
# MODULE - II

## Vector Spaces

A vector space is a collection of objects called vectors that can be manipulated using two fundamental operations addition and Scalar multiplication.

A vector is a quantity having both magnitude and direction. Geometrically a vector $v = (v_1, v_2)$ in a plane can be represented by a line segment with initial point at the origin $(0,0)$ and the terminal point at $(v_1, v_2)$




Vectors in 1-space ($\mathbb{R}$) are real numbers, vectors in 2-space ($\mathbb{R}^2$) are ordered pairs $(v_1, v_2)$, vectors in 3-space ($\mathbb{R}^3$) are ordered 3-tuple $(v_1, v_2, v_3)$ and so on. In general a vector in the n-space ($\mathbb{R}^n$) is an ordered n-tuple $(v_1, v_2, \ldots v_n)$

Let $u = (u_1, u_2, \ldots u_n)$ and $v = (v_1, v_2, \ldots v_n)$ be two vectors in $\mathbb{R}^n$, then vector addition of

$u$ and $v$ is denoted by $u+v$ and is defined as $u+v = (u_1+v_1, u_2+v_2, \dots u_n+v_n)$

Let $u = (u_1, u_2, \dots u_n)$ be a vector in $R^n$ and let $\alpha$ be a scalar, then the <u>scalar multiplication</u> of $\alpha$ and $u$ is given by

$\alpha u = (\alpha u_1, \alpha u_2, \dots \alpha u_n)$

<u>Definition of vector space:</u>

A set $V$ with two operations, vector addition $(+)$ and scalar multiplication $(\cdot)$ is said to be a vector space over real numbers $R$ if it satisfy the following conditions:

Properties of vector addition:

For all $u, v, w \in V$

1) Closure under vector addition: $u+v \in V$
2) Commutative law for addition: $u+v = v+u$
3) Associative law for addition: $u+(v+w) = (u+v)+w$
4) Existence of additive identity: There exists a zero vector in $V$ denoted by $0$ such that for every vector $u \in V$, $u+0 = u$.
5) Existence of additive inverse: For every vector $u \in V$, there exists $-u \in V$ such that $u+(-u) = 0$.

Properties of scalar multiplication:

For all $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$

6) Closure under scalar multiplication: $\alpha u \in V$

7) $\alpha(\beta u) = (\alpha\beta)u$ - Associative law

8) Scalar identity: There exists a scalar identity $1 \in V$ such that $1 \cdot u = u$

9) Distributive law: $(\alpha+\beta)u = \alpha u + \beta u$

10) $\alpha(u+v) = \alpha u + \alpha v$.

Problems

1. Show that $\mathbb{R}^n = \{(v_1, v_2, \ldots v_n), v_i \in \mathbb{R}$ for $i = 1, 2, \ldots n\}$ is a vector space under standard operations.

Ans: 1) Closure under vector addition:

Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then $u = (u_1, u_2, \ldots u_n)$ and $v = (v_1, v_2, \ldots v_n)$

$u+v = (u_1+v_1, u_2+v_2, \ldots u_n+v_n)$, an $n$-tuple with elements real numbers.

$\therefore u+v \in \mathbb{R}^n$

2) commutative law for addition.

Let $u, v \in \mathbb{R}^n$

Then $u+v = (u_1+v_1, u_2+v_2, \ldots u_n+v_n)$

$= (v_1+u_1, v_2+u_2, \ldots v_n+u_n)$

$= (v_1, v_2, \ldots v_n) + (u_1, u_2, \ldots u_n)$

$= v+u.$

3) Associative law for addition.

Let $u, v, w \in \mathbb{R}^n$, then

$$(u+v)+w = (u_1+v_1, u_2+v_2, \dots u_n+v_n) + (w_1, w_2, \dots w_n)$$

$$= ((u_1+v_1)+w_1, (u_2+v_2)+w_2, \dots (u_n+v_n)+w_n)$$

$$= (u_1+(v_1+w_1), u_2+(v_2+w_2), \dots, u_n+(v_n+w_n))$$

$$= (u_1, u_2, \dots u_n) + (v_1+w_1, v_2+w_2, \dots v_n+w_n)$$

$$= u+(v+w)$$

4) Existence of additive identity.

There exist $0 = (0, 0, \dots 0) \in \mathbb{R}^n$ such that

$$u+0 = (u_1, u_2, \dots u_n) + (0, 0, \dots 0)$$

$$= (u_1+0, u_2+0, \dots u_n+0)$$

$$= (u_1, u_2, \dots u_n) = u$$

5) Existence of additive inverse.

There exist $-u = (-u_1, -u_2, \dots -u_n) \in \mathbb{R}^n$

Such that $u+(-u) = (u_1, u_2, \dots u_n) + (-u_1, -u_2, \dots -u_n)$

$$= (u_1-u_1, u_2-u_2, \dots u_n-u_n)$$

$$= (0, 0, \dots 0) = 0$$

6) Closure under scalar multiplication.

For $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$

$\alpha u = (\alpha u_1, \alpha u_2, \dots \alpha u_n) \in \mathbb{R}^n$

7) Associative law.

For $\alpha, \beta \in \mathbb{R}$ and $u \in \mathbb{R}^n$

$$\alpha(\beta u) = \alpha(\beta(u_1, u_2, \dots u_n))$$

$$= \alpha(\beta u_1, \beta u_2, \dots \beta u_n)$$

$$= (\alpha\beta u_1, \alpha\beta u_2, \dots \alpha\beta u_n)$$

$$= \alpha\beta(u_1, u_2, \dots u_n)$$

$$= (\alpha\beta) u$$

8) Scalar identity:

For $u \in \mathbb{R}^n$, there exist $1 \in \mathbb{R}$ such that

$$1 \cdot u = 1 \cdot (u_1, u_2, \dots u_n)$$

$$= (1u_1, 1u_2, \dots 1u_n)$$

$$= (u_1, u_2, \dots u_n) = u.$$

9) Distributive law:

Let $\alpha, \beta \in \mathbb{R}$ and $u \in \mathbb{R}^n$

$$(\alpha+\beta)u = (\alpha+\beta)(u_1, u_2, \dots u_n)$$

$$= ((\alpha+\beta)u_1, (\alpha+\beta)u_2, \dots (\alpha+\beta)u_n)$$

$$= (\alpha u_1 + \beta u_1, \alpha u_2 + \beta u_2, \dots \alpha u_n + \beta u_n)$$

$$= (\alpha u_1, \alpha u_2, \dots \alpha u_n) + (\beta u_1, \beta u_2, \dots \beta u_n)$$

$$= \alpha(u_1, u_2, \dots u_n) + \beta(u_1, u_2, \dots u_n)$$

$$= \alpha u + \beta u.$$

10) For $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$;

$$\alpha(u+v) = \alpha(u_1+v_1,\ u_2+v_2, \ldots, u_n+v_n)$$

$$= (\alpha u_1 + \alpha v_1,\ \alpha u_2 + \alpha v_2, \ldots, \alpha u_n + \alpha v_n)$$

$$= (\alpha u_1, \alpha u_2, \ldots \alpha u_n) + (\alpha v_1, \alpha v_2, \ldots \alpha v_n)$$

$$= \alpha(u_1, u_2, \ldots u_n) + \alpha(v_1, v_2, \ldots v_n)$$

$$= \alpha u + \alpha v.$$

$\therefore \mathbb{R}^n$ is a vector space

2. Check whether the set $\mathbb{R}^2$ of all ordered pairs of real numbers $(x_1, x_2)$ where $x_1, x_2 \in \mathbb{R}$ with addition and scalar multiplication defined as for $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$

$(x_1, x_2) + (y_1, y_2) = (x_1+y_1,\ x_2+y_2)$ and

$c(x_1, x_2) = (cx_1, 0)$

Ans: 1. Closure under vector addition:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, then

$x+y = (x_1+y_1,\ x_2+y_2)$.

$\therefore x+y \in \mathbb{R}^2$

2. Commutative law for addition:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, then

$$x+y = (x_1+y_1,\ x_2+y_2)$$

$$= (y_1+x_1,\ y_2+x_2)$$

$$= (y_1, y_2) + (x_1, x_2) = y+x.$$

3. Associative law for addition:

Let $x=(x_1, x_2)$, $y=(y_1, y_2)$, $z=(z_1, z_2) \in \mathbb{R}^2$, then

$$(x+y)+z = (x_1+y_1, x_2+y_2) + (z_1, z_2)$$

$$= ((x_1+y_1)+z_1, (x_2+y_2)+z_2)$$

$$= (x_1+(y_1+z_1), x_2+(y_2+z_2))$$

$$= (x_1, x_2) + (y_1+z_1, y_2+z_2)$$

$$= x+(y+z)$$

4. Existence of additive identity:

For $x=(x_1, x_2) \in \mathbb{R}^2$, there exists $0=(0,0) \in \mathbb{R}^2$,

such that $x+0 = (x_1, x_2) + (0,0)$

$$= (x_1+0, x_2+0)$$

$$= (x_1, x_2) = x$$

5. Existence of additive inverse:

For $x=(x_1, x_2) \in \mathbb{R}^2$, there exists $-x=(-x_1, -x_2) \in \mathbb{R}^2$,

such that $x+(-x) = (x_1, x_2) + (-x_1, -x_2)$

$$= (x_1-x_1, x_2-x_2)$$

$$= (0,0) = 0.$$

6. Closure under scalar multiplication

For $x=(x_1, x_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$,

$$cx = c(x_1, x_2)$$

$$= (cx_1, 0) \in \mathbb{R}^2$$

7. Associative law

For $c, d \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$

$$c(dx) = c(dx_1, 0)$$
$$= (cdx_1, 0) = cd(x_1, 0)$$

$$(cd)x = cd(x_1, x_2) = (cdx_1, 0)$$
$$= cd(x_1, 0)$$

$\therefore$ From 1 & 2, $c(dx) = (cd)x$

8: Scalar identity

For $1 \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$

$$1 \cdot x = 1(x_1, x_2) = (1x_1, 0)$$
$$= (x_1, 0) \neq x$$

$\therefore$ There is no identity element.

So given $\mathbb{R}^2$ is not a vector space.

3. Show that the set of all $2 \times 3$ matrices over $\mathbb{R}$ is a vector space under standard operations.

Ans: Let $V = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} : a_{ij} \in \mathbb{R}, \text{ for } i = 1, 2 \text{ and } j = 1, 2, 3 \right\}$

1. Closure under vector addition:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \in V$,

then $A+B = \begin{bmatrix} a_{11}+b_{1} & a_{12}+b_{2} & a_{3}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \in V$

2. Commutative law for addition:

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$

$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \end{bmatrix}$

$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = B+A$

3. Associative law for addition.

$(A+B)+C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$

$= \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} & (a_{13}+b_{13})+c_{13} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} & (a_{23}+b_{23})+c_{23} \end{bmatrix}$

$= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & a_{13}+(b_{13}+c_{13}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) & a_{23}+(b_{23}+c_{23}) \end{bmatrix}$

$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} \end{bmatrix}$

$= A+(B+C)$

4. Existence of additive identity

For all $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, there exist $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Such that $A + 0 = A$.

5. Existence of additive inverse

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, there exist $-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$

Such that $A + (-A) = 0$.

6. Closure under scalar multiplication

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in V$ and $\alpha \in R$

$$\alpha A = \alpha \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix} \in V$$

7. Associative law

For $\alpha, \beta \in R$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in V$

$$\alpha(\beta A) = \alpha \begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\beta a_{11} & \alpha\beta a_{12} & \alpha\beta a_{13} \\ \alpha\beta a_{21} & \alpha\beta a_{22} & \alpha\beta a_{23} \end{bmatrix}$$

$$= \alpha\beta \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \alpha\beta(A)$$

8) Scalar identity

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in V$ there exist $1 \in R$

Such that $1 \cdot A = \begin{bmatrix} 1a_{11} & 1a_{12} & 1a_{13} \\ 1a_{21} & 1a_{22} & 1a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$

9. Distributive law

For $\alpha, \beta \in R$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in V$

$$(\alpha+\beta) A = (\alpha+\beta)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} & (\alpha+\beta)a_{13} \\ (\alpha+\beta)a_{21} & (\alpha+\beta)a_{22} & (\alpha+\beta)a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11}+\beta a_{11} & \alpha a_{12}+\beta a_{12} & \alpha a_{13}+\beta a_{13} \\ \alpha a_{21}+\beta a_{21} & \alpha a_{22}+\beta a_{22} & \alpha a_{23}+\beta a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix} + \begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \end{bmatrix}$$

$$= \alpha \cdot A + \beta \cdot A$$

10) For $\alpha \in R$ and $A, B \in V$

$$\alpha(A+B) = \alpha\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(a_{11}+b_{11}) & \alpha(a_{12}+b_{12}) & \alpha(a_{13}+b_{13}) \\ \alpha(a_{21}+b_{21}) & \alpha(a_{22}+b_{22}) & \alpha(a_{23}+b_{23}) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11}+\alpha b_{11} & \alpha a_{12}+\alpha b_{12} & \alpha a_{13}+\alpha b_{13} \\ \alpha a_{21}+\alpha b_{21} & \alpha a_{22}+\alpha b_{22} & \alpha a_{23}+\alpha b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & \alpha b_{12} & \alpha b_{13} \\ \alpha b_{21} & \alpha b_{22} & \alpha b_{23} \end{bmatrix}$$

$$= \alpha A + \alpha B$$

$\therefore$ V is a vector space.

Check whether
4. the set of all $2\times2$ matrices of the form $\begin{bmatrix} x & y \\ 1 & z \end{bmatrix}$, where $x, y, z \in R$ is a vector space or not.

Ans: i) Closure under vector addition

Let $A = \begin{bmatrix} x_1 & y_1 \\ 1 & z_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_2 & y_2 \\ 1 & z_2 \end{bmatrix}$ be elements of given set.

Then $A+B = \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ 1+1 & z_1+z_2 \end{bmatrix}$

$= \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ 2 & z_1+z_2 \end{bmatrix}$, which is not an element of given set

$\therefore$ Given set of matrices is not a vector space.

5. Check whether the set of all $2\times2$ singular matrices over $R$ is a vector space or not.

Ans: Let $V$ be the set of all $2\times2$ singular matrices over $\mathbb{R}$

ie; $V = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix}, x, y, z, w \in R \text{ and } \begin{vmatrix} x & y \\ z & w \end{vmatrix} = 0 \right\}$

Then $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are elements of $V$

Since $|V_1| = 0$ and $|V_2| = 0$

Now $V_1 + V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

But $|V_1+V_2| = 1 \neq 0$

$\therefore V_1 + V_2 \notin V$

So $V$ is not a vector space

6. Check whether the set of all 2x2 non-singular matrices is a vector space or not.

Ans: Let V be the set of all 2x2 non-singular matrices

ie; $V = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x, y, z, w \in R \text{ and } \begin{vmatrix} x & y \\ z & w \end{vmatrix} \neq 0 \right\}$

Then $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are elements

of V, since $|V_1| = 1 \neq 0$ and $|V_2| = 1 \neq 0$.

Now $V_1 + V_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $|V_1 + V_2| = 0$

$\therefore V_1 + V_2 \notin V$

$\therefore$ V is not a vector space.

7. Check whether the set of all 2x2 upper triangular matrices is a vector space or not.

Ans: Let V be the set of all 2x2 upper triangular matrices.

ie; $V = \left\{ \begin{bmatrix} x & y \\ 0 & w \end{bmatrix} : x, y, w \in R \right\}$

i) closure under addition

Let $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 & y_2 \\ 0 & w_2 \end{bmatrix} \in V$,

then $V_1 + V_2 = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ 0 & w_1 + w_2 \end{bmatrix} \in V$

2) Commutative law for addition.

Let $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 & y_2 \\ 0 & w_2 \end{bmatrix} \in V$

then $V_1 + V_2 = \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ 0 & \omega_1+\omega_2 \end{bmatrix}$

$= \begin{bmatrix} x_2+x_1 & y_2+y_1 \\ 0 & \omega_2+\omega_1 \end{bmatrix}$

$= \begin{bmatrix} x_2 & y_2 \\ 0 & \omega_2 \end{bmatrix} + \begin{bmatrix} x_1 & y_1 \\ 0 & \omega_1 \end{bmatrix} = V_2 + V_1$

3. Associative law for addition

$(V_1+V_2)+V_3 = \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ 0 & \omega_1+\omega_2 \end{bmatrix} + \begin{bmatrix} x_3 & y_3 \\ 0 & \omega_3 \end{bmatrix}$

$= \begin{bmatrix} (x_1+x_2)+x_3 & (y_1+y_2)+y_3 \\ 0 & (\omega_1+\omega_2)+\omega_3 \end{bmatrix}$

$= \begin{bmatrix} x_1+(x_2+x_3) & y_1+(y_2+y_3) \\ 0 & \omega_1+(\omega_2+\omega_3) \end{bmatrix}$

$= \begin{bmatrix} x_1 & y_1 \\ 0 & \omega_1 \end{bmatrix} + \begin{bmatrix} x_2+x_3 & y_2+y_3 \\ 0 & \omega_2+\omega_3 \end{bmatrix}$

$= V_1 + (V_2+V_3)$

4) Existence of additive identity

For all $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & \omega_1 \end{bmatrix}$ there exist $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$

Such that $V_1 + 0 = V_1$

5) Existence of additive inverse

For all $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & \omega_1 \end{bmatrix}$ there exist $-V_1 = \begin{bmatrix} -x_1 & -y_1 \\ 0 & -\omega_1 \end{bmatrix}$

in $V$ such that $V_1 + (-V_1) = 0$

6) Closure under scalar multiplication

For $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix} \in V$ and $\alpha \in R$

$$\alpha V_1 = \begin{bmatrix} \alpha x_1 & \alpha y_1 \\ 0 & \alpha w_1 \end{bmatrix} \in V$$

7) Associative law

For $\alpha, \beta \in R$ and $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix} \in V$

$$\alpha(\beta \cancel{x_1}) = \alpha \begin{bmatrix} \beta x_1 & \beta y_1 \\ 0 & \beta w_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\beta x_1 & \alpha\beta y_1 \\ 0 & \alpha\beta w_1 \end{bmatrix}$$

$$= \alpha\beta \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix} = (\alpha\beta) V_1$$

8) Scalar identity

For $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix} \in V$ there exists $1 \in R$

such that $1 \cdot V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix} = V_1$

9) Distributive law

For $\alpha, \beta \in R$ and $V_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & w_1 \end{bmatrix} \in V$

$$(\alpha+\beta) V_1 = \begin{bmatrix} (\alpha+\beta)x_1 & (\alpha+\beta)y_1 \\ 0 & (\alpha+\beta)w_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 + \beta x_1 & \alpha y_1 + \beta y_1 \\ 0 & \alpha w_1 + \beta w_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 & \alpha y_1 \\ 0 & \alpha w_1 \end{bmatrix} + \begin{bmatrix} \beta x_1 & \beta y_1 \\ 0 & \beta w_1 \end{bmatrix}$$

$$= \alpha V_1 + \beta V_1$$

10) For $\alpha \in R$ and $V_1, V_2 \in V$

$$\alpha(V_1+V_2) = \alpha\begin{bmatrix} x_1+x_2 & y_1+y_2 \\ 0 & \omega_1+\omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(x_1+x_2) & \alpha(y_1+y_2) \\ 0 & \alpha(\omega_1+\omega_2) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1+\alpha x_2 & \alpha y_1+\alpha y_2 \\ 0 & \alpha\omega_1+\alpha\omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 & \alpha y_1 \\ 0 & \alpha\omega_1 \end{bmatrix} + \begin{bmatrix} \alpha x_2 & \alpha y_2 \\ 0 & \alpha\omega_2 \end{bmatrix}$$

$$= \alpha V_1 + \alpha V_2$$

$\therefore$ $V$ is a vector space.

## Subspaces

A nonempty subset W of a vector space V is called a subspace of V if W itself is a vector space under the same operations of vector addition and scalar multiplication defined in V.

### Theorem

If W is a non empty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold:

1) If $u, v \in W$, then $u+v \in W$

2) If $u \in W$ and $c$ is any scalar, then $cu \in W$

### Problems

1. Let W be the set of all $2\times 2$ symmetric matrices. Show that W is a subspace of $M_{2,2}$ over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

Ans: A is symmetric if $A = A^T$.

Given $W = \{A \in M_{2,2} : A^T = A\}$

Let $A, B \in W$. Then $A = A^T$ and $B = B^T$

Now $(A+B)^T = A^T + B^T = A + B$

$\therefore A+B \in W$

Let $A \in W$ and $\alpha$ is any scalar, then

$(\alpha A)^T = \alpha A^T = \alpha A$

$\therefore \alpha A \in W$. $\therefore$ W is a subspace of $M_{2,2}$

2. Let $W$ be the set of all $2\times2$ singular matrices over $\mathbb{R}$. Show that $W$ is not a subspace of $M_{2,2}$ over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

Ans: A square matrix '$A$' is singular if $|A|=0$

Given $W=\{A\in M_{2,2} : |A|=0\}$

Let $V_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $V_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ are elements

of $W$, since $\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}=0$ and $\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}=0$

But $V_1+V_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$ since $\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1\neq0$

$\therefore W$ is not a subspace of $M_{2,2}$.

3. Show that $W=\{(x, mx): x\in\mathbb{R}$ and $m$, a scalar$\}$ is a subspace of $\mathbb{R}^2$.

Ans: $W=\{(x,mx): x\in\mathbb{R}$ and $m$, a scalar$\}$

Let $w_1=(x_1, mx_1)$ and $w_2=(x_2, mx_2)\in W$ and $\alpha\in\mathbb{R}$

Then $w_1+w_2=(x_1+x_2,\ mx_1+mx_2)$

$=(x_1+x_2,\ m(x_1+x_2))\in W$

$\alpha w_1=\alpha(x_1, mx_1)=(\alpha x_1, m\alpha x_1)\in W$

$\therefore W$ is a subspace of $\mathbb{R}^2$

4. Show that $W = \{(x, 2x+1) : x \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$ under standard operations.

Ans: $W = \{(x, 2x+1); x \in \mathbb{R}\}$

Let $w_1 = (x_1, 2x_1+1)$, $w_2 = (x_2, 2x_2+1) \in W$

Then $w_1 + w_2 = (x_1 + x_2, 2x_1 + 1 + 2x_2 + 1)$

$= (x_1 + x_2, 2x_1 + 2x_2 + 2)$

$= (x_1 + x_2, 2(x_1 + x_2) + 2) \notin W$

$\therefore$ W is not a subspace of $\mathbb{R}^2$.

5. Check whether $W = \{(x, x+3) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ under standard operations.

Ans: $W = \{(x, x+3) : x \in \mathbb{R}\}$

Let $w_1 = (x_1, x_1+3)$ and $w_2 = (x_2, x_2+3) \in W$

Then $w_1 + w_2 = (x_1 + x_2, x_1 + 3 + x_2 + 3)$

$= (x_1 + x_2, x_1 + x_2 + 6) \notin W$

$\therefore$ W is not a subspace of $\mathbb{R}^2$.

6. Show that $W = \{(x, y, y-2x) : x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ under standard operations.

Ans: Given $W = \{(x, y, y-2x) : x, y \in \mathbb{R}\}$

Let $w_1 = (x_1, y_1, y_1 - 2x_1)$ and $w_2 = (x_2, y_2, y_2 - 2x_2) \in W$

Then $w_1 + w_2 = (x_1 + x_2, y_1 + y_2, y_1 - 2x_1 + y_2 - 2x_2)$

$= (x_1 + x_2, y_1 + y_2, y_1 + y_2 - 2(x_1 + x_2)) \in W$

Let $\alpha \in \mathbb{R}$, then $\alpha w_1 = \alpha(x_1, y_1, y_1, -2x_1)$

$$= (\alpha x_1, \alpha y_1, \alpha y_1, -2\alpha x_1) \in W$$

$\therefore$ W is a subspace of $\mathbb{R}^3$.

7. Check whether set of all points on the plane $2x - y + 3z = 0$ is a subspace of $\mathbb{R}^3$ under Standard operations.

Ans: Given the plane $2x - y + 3z = 0$

$$\Rightarrow -y = -2x - 3z$$

$$\Rightarrow y = 2x + 3z$$

$\therefore W = \{(x, 2x+3z, z) : x, z \in \mathbb{R}\}$

Let $w_1 = (x_1, 2x_1 + 3z_1, z_1)$ and $w_2 = (x_2, 2x_2 + 3z_2, z_2) \in W$

Then $w_1 + w_2 = (x_1 + x_2, 2x_1 + 3z_1 + 2x_2 + 3z_2, z_1 + z_2)$

$$= (x_1 + x_2, 2(x_1 + x_2) + 3(z_1 + z_2), z_1 + z_2) \in W$$

Let $\alpha \in \mathbb{R}$, then $\alpha w_1 = \alpha(x_1, 2x_1 + 3z_1, z_1)$

$$= (\alpha x_1, 2\alpha x_1 + 3\alpha z_1, \alpha z_1) \in W$$

$\therefore$ W is a subspace of $\mathbb{R}^3$.

8. Check whether set of all points on the plane $x - 2y + 3z = 1$ is a subspace of $\mathbb{R}^3$ under standard operations.

Ans: Given the plane $x - 2y + 3z = 1$

$$\Rightarrow x = 1 + 2y - 3z$$

$\therefore W = \{(1+2y-3z, y, z) : y, z \in \mathbb{R}\}$

Let $w_1 = (1+2y_1-3z_1, y_1, z_1)$ and $w_2 = (1+2y_2-3z_2, y_2, z_2)$ $\in W$

Then $w_1 + w_2 = (1+2y_1-3z_1+1+2y_2-3z_2, y_1+y_2, z_1+z_2)$

$= (2+2(y_1+y_2)-3(z_1+z_2), y_1+y_2, z_1+z_2) \notin W$

$\therefore$ W is not a subspace of $\mathbb{R}^3$.

9. Check whether W = set of all $3\times2$ matrices of the form $\begin{bmatrix} a & b-1 \\ a+b & 0 \\ b & c \end{bmatrix}$ is a subspace of $M_{3,2}$ under standard matrix addition and scalar multiplication.

Ans: Let $w_1 = \begin{bmatrix} a_1 & b_1-1 \\ a_1+b_1 & 0 \\ b_1 & c_1 \end{bmatrix}$, $w_2 = \begin{bmatrix} a_2 & b_2-1 \\ a_2+b_2 & 0 \\ b_2 & c_2 \end{bmatrix} \in W$

Then $w_1 + w_2 = \begin{bmatrix} a_1+a_2 & b_1-1+b_2-1 \\ a_1+b_1+a_2+b_2 & 0 \\ b_1+b_2 & c_1+c_2 \end{bmatrix}$

$= \begin{bmatrix} a_1+a_2 & b_1+b_2-2 \\ a_1+a_2+b_1+b_2 & 0 \\ b_1+b_2 & c_1+c_2 \end{bmatrix} \notin W$

$\therefore$ W is not a subspace of $M_{3,2}$.

## Linear combination of vectors

A linear combination of $m$ vectors $v_1, v_2, \ldots v_m$ is defined as $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$ where $\alpha_1, \alpha_2, \ldots \alpha_m$ are some scalars.

### Problems

1. Write the vector $(6,-1,-3)$ as a linear combination of the vectors in the set $S = \{(1,-1,2), (2,1,-1), (-1,2,3)\}$.

Ans: we have to find scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$(6,-1,-3) = \alpha_1(1,-1,2) + \alpha_2(2,1,-1) + \alpha_3(-1,2,3) \quad (1)$$

$$= (\alpha_1, -\alpha_1, 2\alpha_1) + (2\alpha_2, \alpha_2, -\alpha_2) + (-\alpha_3, 2\alpha_3, 3\alpha_3)$$

$$= (\alpha_1 + 2\alpha_2 - \alpha_3, -\alpha_1 + \alpha_2 + 2\alpha_3, 2\alpha_1 - \alpha_2 + 3\alpha_3)$$

$$\Rightarrow \alpha_1 + 2\alpha_2 - \alpha_3 = 6$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = -1$$

$$2\alpha_1 - \alpha_2 + 3\alpha_3 = -3$$

Now solving these equations using Gauss Elimination method,

Augmented matrix $= \begin{bmatrix} 1 & 2 & -1 & 6 \\ -1 & 1 & 2 & -1 \\ 2 & -1 & 3 & -3 \end{bmatrix}$ $\quad R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 3 & 1 & 5 \\ 0 & -5 & 5 & -15 \end{bmatrix} \quad R_2 \to \frac{R_2}{3}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & -5 & 5 & -15 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 20/3 & -20/3 \end{bmatrix} \quad R_3 \to R_3 \times \frac{3}{20}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\therefore \alpha_1 + 2\alpha_2 - \alpha_3 = 6$

$\alpha_2 + \frac{1}{3}\alpha_3 = \frac{5}{3}$

$\alpha_3 = -1$

$\therefore \alpha_2 + \frac{1}{3} \times -1 = \frac{5}{3} \Rightarrow \alpha_2 - \frac{1}{3} = \frac{5}{3} \Rightarrow \alpha_2 = \frac{5}{3} + \frac{1}{3}$

$= \frac{6}{3} = 2$

$\therefore \alpha_1 + 2 \times 2 - -1 = 6 \Rightarrow \alpha_1 + 4 + 1 = 6$

$\Rightarrow \alpha_1 + 5 = 6 \Rightarrow \alpha_1 = 6 - 5 = 1$

$\therefore$ ① $\Rightarrow (6, -1, -3) = 1(1, -1, 2) + 2(2, 1, -1) - 1(-1, 2, 3)$

2. Check whether the vector $(1, -2, 2)$ can be written as a linear combination of the vectors in the set $\{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$

Ans: We have to find scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$(1, -2, 2) = \alpha_1(1, 2, 3) + \alpha_2(0, 1, 2) + \alpha_3(-1, 0, 1)$ — ①

$= (\alpha_1, 2\alpha_1, 3\alpha_1) + (0, \alpha_2, 2\alpha_2) + (-\alpha_3, 0, \alpha_3)$

$= (\alpha_1 - \alpha_3, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 + \alpha_3)$

$\Rightarrow \alpha_1 - \alpha_3 = 1$

$2\alpha_1 + \alpha_2 = -2$

$3\alpha_1 + 2\alpha_2 + \alpha_3 = 2$

To solve using Gauss Elimination method,

Augmented matrix $= \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{bmatrix}$ $\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 2 & 4 & -1 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

$\therefore$ Rank of $[AB] \neq$ Rank of $A$

$\therefore$ There is no solution.

So we cannot find $\alpha_1, \alpha_2, \alpha_3$ which satisfies ①

$\therefore$ The vector $(1, -2, 2)$ can not be written as a linear combination of given vectors.

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3. Check whether the vector $(1, 2, 1)$ can be written as a linear combination of the vectors in the set $\{(1, -1, -3), (0, 3, 2), (-1, 0, 2)\}$.

Ans: We have to find scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$(1, 2, 1) = \alpha_1(1, -1, -3) + \alpha_2(0, 3, 2) + \alpha_3(-1, 0, 2)$ — ①

$= (\alpha_1, -\alpha_1, -3\alpha_1) + (0, 3\alpha_2, 2\alpha_2) + (-\alpha_3, 0, 2\alpha_3)$

$= (\alpha_1 - \alpha_3, -\alpha_1 + 3\alpha_2, -3\alpha_1 + 2\alpha_2 + 2\alpha_3)$

$\Rightarrow \alpha_1 - \alpha_3 = 1$

$-\alpha_1 + 3\alpha_2 = 2$

$-3\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$

To solve using Gauss Elimination method,

augmented matrix = $\begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 3 & 0 & 2 \\ -3 & 2 & 2 & 1 \end{bmatrix}$ $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 3R_1$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 3 \\ 0 & 2 & -1 & 4 \end{bmatrix}$ $R_2 \to \frac{R_2}{3}$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1/3 & 1 \\ 0 & 2 & -1 & 4 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1/3 & 1 \\ 0 & 0 & -1/3 & 2 \end{bmatrix}$ $R_3 \to R_3 \times -3$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1/3 & 1 \\ 0 & 0 & 1 & -6 \end{bmatrix}$

$\therefore \alpha_1 - \alpha_3 = 1$

$\alpha_2 - \frac{1}{3}\alpha_3 = 1$

$\alpha_3 = -6$

$\therefore \alpha_2 - \frac{1}{3}\alpha_3 = 1 \Rightarrow \alpha_2 - \frac{1}{3} \times -6 = 1 \Rightarrow \alpha_2 + 2 = 1$

$\Rightarrow \alpha_2 = 1 - 2 = -1$

$\alpha_1 - -6 = 1 \Rightarrow \alpha_1 + 6 = 1$

$\Rightarrow \alpha_1 = 1 - 6 = -5$

$\therefore$ ① $\Rightarrow \underline{(1,2,1) = -5(1,-1,-3) - 1(0,3,2) - 6(-1,0,2)}$

## Spanning sets

Let $S = \{v_1, v_2, \ldots v_k\}$ be a subset of a vector space $V$. Then the set $S$ is called a spanning set of $V$ if every vector in $V$ is a linear combination of vectors in $S$.

**Note**

The system $AX = B$ has a unique solution if $|A| \neq 0$.

**Problems**

1. Show that $S = \{(1,2), (0,1)\}$ is a spanning set of $\mathbb{R}^2$.

Ans: Let $(x, y)$ be an arbitrary vector in $\mathbb{R}^2$. If $S$ spans $\mathbb{R}^2$, then there exist scalars $\alpha_1, \alpha_2$ such that

$$(x, y) = \alpha_1 (1, 2) + \alpha_2 (0, 1) \quad \text{— ①}$$

$$= (\alpha_1, 2\alpha_1) + (0, \alpha_2)$$

$$= (\alpha_1, 2\alpha_1 + \alpha_2)$$

$\Rightarrow \alpha_1 = x$

$2\alpha_1 + \alpha_2 = y \Rightarrow 2x + \alpha_2 = y$

$\Rightarrow \alpha_2 = y - 2x$

$\therefore$ ① $\Rightarrow (x, y) = x(1,2) + (y - 2x)(0,1)$.

So $S$ spans $\mathbb{R}^2$.

2. Check whether the set $S = \{(1,2,3), (1,-1,0), (0,0,1)\}$ is a spanning set of $\mathbb{R}^3$.

Ans: Let $(x, y, z)$ be an arbitrary vector in $\mathbb{R}^3$.

If $S$ spans $\mathbb{R}^3$, then there exists scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$(x, y, z) = \alpha_1 (1, 2, 3) + \alpha_2 (1, -1, 0) + \alpha_3 (0, 0, 1) \quad \text{--- (1)}$$

$$= (\alpha_1, 2\alpha_1, 3\alpha_1) + (\alpha_2, -\alpha_2, 0) + (0, 0, \alpha_3)$$

$$= (\alpha_1 + \alpha_2, \; 2\alpha_1 - \alpha_2, \; 3\alpha_1 + \alpha_3)$$

$\Rightarrow \alpha_1 + \alpha_2 = x$

$2\alpha_1 - \alpha_2 = y$

$3\alpha_1 + \alpha_3 = z$

Determinant of the co-efficient matrix is

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = -3 \neq 0$$

$\therefore$ This linear system of equations has a unique solution. Hence there exists $\alpha_1, \alpha_2, \alpha_3$ which satisfies equation (1).

So $S$ spans $\mathbb{R}^3$.

3. Check whether the set $S = \{(1, -1, 3), (1, -1, 0), (0, 0, 1)\}$ is a spanning set of $\mathbb{R}^3$.

Ans: Let $(x, y, z)$ be an arbitrary vector in $\mathbb{R}^3$.

If $S$ spans $\mathbb{R}^3$, then there exists scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$(x, y, z) = \alpha_1 (1, -1, 3) + \alpha_2 (1, -1, 0) + \alpha_3 (0, 0, 1) \quad \text{--- (1)}$$

$$= (\alpha_1, -\alpha_1, 3\alpha_1) + (\alpha_2, -\alpha_2, 0) + (0, 0, \alpha_3)$$

$= (\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2, 3\alpha_1 + \alpha_3)$

$\Rightarrow \alpha_1 + \alpha_2 = x$

$-\alpha_1 - \alpha_2 = y$

$3\alpha_1 + \alpha_3 = z$

Here determinant of the coefficient matrix is

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 0$$

If $|A| = 0$, then the linear system $AX = B$ may have infinitely many solutions or have no solutions at all.

Let us solve the system by Gauss elimination method

Augmented matrix $= \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ -1 & -1 & 0 & y \\ 3 & 0 & 1 & z \end{array}\right]$ $R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 0 & 0 & y+x \\ 0 & -3 & 1 & z-3x \end{array}\right] \quad R_3 \rightarrow \frac{R_3}{-3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 0 & 0 & y+x \\ 0 & 1 & -1/3 & \frac{z-3x}{3} \end{array}\right]$$

$\therefore$ Rank of $[AB] = 3$

Rank of $A = 2$.

So the system has no solution

$\therefore$ $S$ doesnot spans $\mathbb{R}^3$.

# Linearly independent and dependent vectors

A set of vectors $\{v_1, v_2, \ldots v_m\}$ in a vector space $V$ is said to be linearly independent if the vector equation $c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0$ has only the trivial solution $c_1 = 0, c_2 = 0, \ldots, c_m = 0$.

If there is a solution for the equation with some $c_i \neq 0$, then the set of vectors is said to be linearly dependent.

## Problems

1. Check whether the set of vectors $\{(1,2,0), (2,5,1), (-5,1,2)\}$ is linearly independent or not.

Ans: Consider the equation,

$$c_1(1,2,0) + c_2(2,5,1) + c_3(-5,1,2) = (0,0,0)$$

$$\Rightarrow (c_1, 2c_1, 0) + (2c_2, 5c_2, c_2) + (-5c_3, c_3, 2c_3) = (0,0,0)$$

$$\Rightarrow (c_1 + 2c_2 - 5c_3,\ 2c_1 + 5c_2 + c_3,\ c_2 + 2c_3) = (0,0,0)$$

$$\Rightarrow c_1 + 2c_2 - 5c_3 = 0$$

$$2c_1 + 5c_2 + c_3 = 0$$

$$c_2 + 2c_3 = 0$$

Augmented matrix $= \begin{bmatrix} 1 & 2 & -5 & 0 \\ 2 & 5 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & -9 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{-9}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ $c_1 + 2c_2 - 5c_3 = 0$

$c_2 + 11c_3 = 0$

$c_3 = 0 \Rightarrow c_3 = 0$

$\therefore$ $c_2 + (11 \times 0) = 0 \Rightarrow c_2 = 0$

$c_1 + 0 - 0 = 0 \Rightarrow c_1 = 0$

Since $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, the given vectors are linearly independent.

2. Check whether the set of vectors $\{(1, 3, 2), (2, -1, 4), (3, 2, 6)\}$ is linearly independent or not.

Ans: Consider the equation

$c_1(1, 3, 2) + c_2(2, -1, 4) + c_3(3, 2, 6) = (0, 0, 0)$

$\Rightarrow (c_1, 3c_1, 2c_1) + (2c_2, -c_2, 4c_2) + (3c_3, 2c_3, 6c_3) = (0, 0, 0)$

$\Rightarrow (c_1 + 2c_2 + 3c_3, 3c_1 - c_2 + 2c_3, 2c_1 + 4c_2 + 6c_3) = (0, 0, 0)$

$\Rightarrow c_1 + 2c_2 + 3c_3 = 0$

$3c_1 - c_2 + 2c_3 = 0$

$2c_1 + 4c_2 + 6c_3 = 0$

Augmented matrix = $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & -1 & 2 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to \frac{R_2}{-7}$

$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore c_1 + 2c_2 + 3c_3 = 0$

$c_2 + c_3 = 0$

Here rank = 2 and no. of unknowns = 3. So we have to assign arbitrary values to 3-2=1 variable

Let $c_3 = a$

$\therefore c_2 + a = 0 \Rightarrow c_2 = -a$

$c_1 - 2a + 3a = 0 \Rightarrow c_1 + a = 0 \Rightarrow c_1 = -a$

Let $a = 1$, then $c_1 = -1$, $c_2 = -1$, $c_3 = 1$

$\therefore$ The given vectors are linearly dependent.

3. Show that the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent where $v_1 = (1, -1, 3, 1)$, $v_2 = (-1, 1, 2, 4)$; $v_3 = (1, -2, -2, 3)$

Ans: Consider the equation

$c_1(1, -1, 3, 1) + c_2(-1, 1, 2, 4) + c_3(1, -2, -2, 3) = (0,0,0,0)$

$\Rightarrow (c_1, -c_1, 3c_1, c_1) + (-c_2, c_2, 2c_2, 4c_2) + (c_3, -2c_3, -2c_3, 3c_3) = (0,0,0,0)$

$\Rightarrow (c_1 - c_2 + c_3, -c_1 + c_2 - 2c_3, 3c_1 + 2c_2 - 2c_3, c_1 + 4c_2 + 3c_3) = (0,0,0,0)$

$$\Rightarrow \begin{aligned} c_1 - c_2 + c_3 &= 0 \\ -c_1 + c_2 - 2c_3 &= 0 \\ 3c_1 + 2c_2 - 2c_3 &= 0 \\ c_1 + 4c_2 + 3c_3 &= 0 \end{aligned}$$

Augmented matrix $= \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 3 & 2 & -2 & 0 \\ 1 & 4 & 3 & 0 \end{bmatrix}$ $\begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \quad R_2 \to \frac{R_2}{-1}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 5R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{5}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - 5R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $[AB] = 3 =$ Rank of A

$$\therefore \begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_3 &= 0 \\ c_2 &= 0 \end{aligned}$$

$\therefore c_1 - 0 + 0 = 0 \Rightarrow c_1 = 0$

Since $c_1=0$, $c_2=0$, $c_3=0$, given vectors are linearly independent.

4. Check whether the set of vectors

$$S=\left\{\begin{bmatrix}1&2\\0&1\end{bmatrix},\begin{bmatrix}1&0\\3&-1\end{bmatrix},\begin{bmatrix}3&0\\-1&2\end{bmatrix}\right\}$$ in $M_{2,2}$ is linearly independent or not.

Ans: Consider the equation

$$c_1\begin{bmatrix}1&2\\0&1\end{bmatrix}+c_2\begin{bmatrix}1&0\\3&-1\end{bmatrix}+c_3\begin{bmatrix}3&0\\-1&2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}c_1&2c_1\\0&c_1\end{bmatrix}+\begin{bmatrix}c_2&0\\3c_2&-c_2\end{bmatrix}+\begin{bmatrix}3c_3&0\\-c_3&2c_3\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}c_1+c_2+3c_3&2c_1\\3c_2-c_3&c_1-c_2+2c_3\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow c_1+c_2+3c_3=0$$
$$2c_1=0$$
$$3c_2-c_3=0$$
$$c_1-c_2+2c_3=0$$

Augmented matrix $=\begin{bmatrix}1&1&3&0\\2&0&0&0\\0&3&-1&0\\1&-1&2&0\end{bmatrix}$ $R_2\to R_2-2R_1$, $R_4\to R_4-R_1$

$$\sim\begin{bmatrix}1&1&3&0\\0&-2&-6&0\\0&3&-1&0\\0&-2&-1&0\end{bmatrix}$$ $R_2\to\frac{R_2}{-2}$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{-10}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \quad R_4 \to R_4 - 5R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[AB] = 3 =$ Rank of $A$

$$c_1 + c_2 + 3c_3 = 0$$
$$c_2 + 3c_3 = 0$$
$$c_3 = 0$$

$$c_2 + 0 = 0 \Rightarrow c_2 = 0$$

$$c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$$

Since $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, given vectors are linearly independent.

## Note

If $A$ is a square matrix, the system $AX=0$ has non-trivial solutions if and only if $|A|=0$.

For which value of $\lambda$ do the set of vectors $\{(\lambda,-1,-1),(-1,\lambda,-1),(-1,-1,\lambda)\}$ form a linearly dependent set in $\mathbb{R}^3$?

Ans: consider the equation

$$\alpha_1(\lambda,-1,-1)+\alpha_2(-1,\lambda,-1)+\alpha_3(-1,-1,\lambda)=0$$

$$\Rightarrow (\alpha_1\lambda-\alpha_2-\alpha_3,\ -\alpha_1+\lambda\alpha_2-\alpha_3,\ -\alpha_1-\alpha_2+\lambda\alpha_3)=(0,0,0).$$

$$\therefore \alpha_1\lambda-\alpha_2-\alpha_3=0$$
$$-\alpha_1+\lambda\alpha_2-\alpha_3=0$$
$$-\alpha_1-\alpha_2+\lambda\alpha_3=0$$

This system is of the form $AX=0$, where

$$A=\begin{bmatrix}\lambda & -1 & -1\\ -1 & \lambda & -1\\ -1 & -1 & \lambda\end{bmatrix}$$ and this system has non-trivial solution if $|A|=0$

$$\Rightarrow \lambda(\lambda^2-1)+1(-\lambda-1)-1(1+\lambda)=0$$

$$\Rightarrow \lambda^3-\lambda-\lambda-1-1-\lambda=0$$

$$\Rightarrow \lambda^3-3\lambda-2=0$$

$$\Rightarrow \lambda=-1,2$$

## Basis and Dimension.

A set of vectors $S = \{v_1, v_2, \dots v_k\}$ in a vector space $V$ is a basis for $V$ if $S$ satisfies the following conditions.

1) $S$ spans $V$
2) $S$ is linearly independent.

The number of elements in the basis of a vector space $V$ is called the dimension of $V$ and is denoted as $\dim(V)$.

Note

1) If a vector space has a basis consisting of only finitely many vectors, then it is called a finite dimensional vector space and if the basis contains infinitely many vectors, then it is called infinite dimensional.

2) The standard basis of $\mathbb{R}^2$ is $\{(1,0), (0,1)\}$, the standard basis of $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$

In general the standard basis of $\mathbb{R}^n$ is

$$\{(1,0,0,\dots 0), (0,1,0,\dots 0), \dots (0,0,0,\dots 1)\}$$

3) If $S = \{v_1, v_2, \dots v_k\}$ is a basis for a vector space $V$, then every vector in $V$ can be

written in one and only one way as a linear combination of vectors in S.

4) Any basis of a vector space V has same number of elements.

5) If S is a basis for a vector space V containing n vectors, then every set containing more than n vectors in V is linearly dependent.

6) [Dim(V) = 0 when $V = \{0\}$] Here V contains only the zero vector and it cannot be included in any basis of V since a set containing 0 is not linearly independent. So we cannot form a basis for V.

7) $\text{Dim}(\mathbb{R}) = 1$

$\text{Dim}(\mathbb{R}^2) = 2$

$\text{Dim}(\mathbb{R}^3) = 3$

In general $\dim(\mathbb{R}^n) = n$

$\text{Dim}(M_{n,n}) = n^2$ and $\dim(M_{m,n}) = mn$.

8) Let V be a vector space having dimension n. Then the following holds.

1) If S is a subset of V containing n vectors and if S is linearly independent, then S is a basis for V.

2) If S is a subset of V containing n vectors and if S spans V, then S is a basis for V.

1. Show that the set of vectors $S = \{(1,2,0), (2,5,1), (-5,1,2)\}$ forms a basis of $\mathbb{R}^3$.

Ans: To show $S$ spans $\mathbb{R}^3$, let $(x, y, z)$ be an arbitrary vector in $\mathbb{R}^3$. If $S$ spans $\mathbb{R}^3$, there exist scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$(x, y, z) = \alpha_1(1,2,0) + \alpha_2(2,5,1) + \alpha_3(-5,1,2) \quad \text{--- (1)}$$

$$= (\alpha_1 + 2\alpha_2 - 5\alpha_3,\ 2\alpha_1 + 5\alpha_2 + \alpha_3,\ 0\alpha_1 + 1\alpha_2 + 2\alpha_3)$$

$$\Rightarrow \alpha_1 + 2\alpha_2 - 5\alpha_3 = x$$
$$2\alpha_1 + 5\alpha_2 + \alpha_3 = y$$
$$0\alpha_1 + 1\alpha_2 + 2\alpha_3 = z$$

, which is of the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$|A| = 1(10-1) - 2(4-0) - 5(2-0)$$

$$= 9 - 8 - 10 = 9 - 18 = -9 \neq 0$$

$\therefore$ There exist $\alpha_1, \alpha_2, \alpha_3$ satisfies equation (1).

So $S$ spans $\mathbb{R}^3$.

(1) To show $S$ is linearly independent, consider the equation,

$$\alpha_1(1,2,0) + \alpha_2(2,5,1) + \alpha_3(-5,1,2) = (0,0,0) \quad \text{--- (2)}$$

$$\Rightarrow (\alpha_1 + 2\alpha_2 - 5\alpha_3,\ 2\alpha_1 + 5\alpha_2 + \alpha_3,\ 0\alpha_1 + \alpha_2 + 2\alpha_3) = (0,0,0)$$

$$\Rightarrow \alpha_1 + 2\alpha_2 - 5\alpha_3 = 0,\quad 2\alpha_1 + 5\alpha_2 + \alpha_3 = 0,\quad 0\alpha_1 + \alpha_2 + 2\alpha_3 = 0$$

which is of the form $AX=0$, with

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$|A| = -9 \neq 0$

$\therefore$ The system has unique solution and since $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is a solution, it is the only solution of equation 2.

$\therefore$ S is linearly independent.

~~Since S spans $\mathbb{R}^3$ and S is linearly independent, S is a basis of $\mathbb{R}^3$.~~

Since $\dim(\mathbb{R}^3) = 3$ and S is a linearly independent set with 3 elements, S is a basis of $\mathbb{R}^3$.

2. Show that $\{(1,2,-3), (0,1,-2), (1,1,0)\}$ form a basis of $\mathbb{R}^3$.

Ans: Let $S = \{(1,2,-3), (0,1,-2), (1,1,0)\}$

To check whether S is linearly independent, consider the equation

$$\alpha_1(1,2,-3) + \alpha_2(0,1,-2) + \alpha_3(1,1,0) = 0 \quad \text{— 1}$$

$\Rightarrow (\alpha_1 + 0\alpha_2 + \alpha_3,\ 2\alpha_1 + 1\alpha_2 + 1\alpha_3,\ -3\alpha_1 - 2\alpha_2 + 0\alpha_3) = (0,0,0)$

$\Rightarrow \alpha_1 + 0\alpha_2 + \alpha_3 = 0$

$2\alpha_1 + 1\alpha_2 + 1\alpha_3 = 0$ , which is a system of

$-3\alpha_1 - 2\alpha_2 + 0\alpha_3 = 0$ equations of the form $AX=0$

with $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 0 \end{bmatrix}$

$|A| = 1(0 - -2) + 1(-4 - -3)$

$= 2 + -1 = 1 \neq 0$

Since $|A| \neq 0$, the system has unique solution and since $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is a solution, it is the only solution of ①

$\therefore S$ is linearly independent.

Since $\dim(\mathbb{R}^3) = 3$ and $S$ is a linearly independent set with 3 elements, $S$ is a basis of $\mathbb{R}^3$.

3. Show that $\{(1,0,1), (0,1,1), (1,1,0)\}$ forms a basis of $\mathbb{R}^3$.

4. Show that $S = \{M_1, M_2, M_3, M_4\} \subset M_{2,2}$ given by

$M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ forms a basis of $M_{2,2}$.

Ans: To check whether $S$ is linearly independent, consider the equation

$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = 0$ —— ①

$\Rightarrow \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \alpha_1+\alpha_2+0\alpha_3+1\alpha_4 & \alpha_1-\alpha_2-\alpha_3+0\alpha_4 \\ \alpha_1+0\alpha_2+1\alpha_3+0\alpha_4 & \alpha_1+0\alpha_2+0\alpha_3+0\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \alpha_1+\alpha_2+0\alpha_3+1\alpha_4=0$

$\alpha_1-\alpha_2-\alpha_3+0\alpha_4=0$

$\alpha_1+0\alpha_2+1\alpha_3+0\alpha_4=0$ , which is of the

$\alpha_1+0\alpha_2+0\alpha_3+0\alpha_4=0$ form $AX=0$ with

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore |A| = 1\begin{vmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1(0) + 1(-1) = -1$$

$\therefore |A| \neq 0$

$\therefore$ The system has unique solution and since $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ is a solution, it is the only solution of equation ①

$\therefore S$ is linearly independent

Since $\dim(M_{2,2})=4$ and $S$ is a linearly independent set with 4 vectors, $S$ is a basis of $M_{2,2}$

5. Show that $S=\{M_1, M_2, M_3, M_4\} \subset M_{2,2}$ given by $M_1=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $M_2=\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $M_3=\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, $M_4=\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ forms a basis of $M_{2,2}$.

6 Check whether $S = \{(2,0,-1,4), (3,1,-2,0), (1,-1,0,8)\}$ is a basis of $\mathbb{R}^4$.

Ans: To check linear independence, consider the equation $\alpha_1(2,0,-1,4) + \alpha_2(3,1,-2,0) + \alpha_3(1,-1,0,8) = 0$

$$\Rightarrow (2\alpha_1 + 3\alpha_2 + \alpha_3,\ 0\alpha_1 + 1\alpha_2 - \alpha_3,\ -\alpha_1 - 2\alpha_2 + 0\alpha_3,\ 4\alpha_1 + 0\alpha_2 + 8\alpha_3) = (0,0,0,0)$$

$$\Rightarrow 2\alpha_1 + 3\alpha_2 + \alpha_3 = 0$$

$$0\alpha_1 + 1\alpha_2 - \alpha_3 = 0$$

$$-\alpha_1 - 2\alpha_2 + 0\alpha_3 = 0$$

$4\alpha_1 + 0\alpha_2 + 8\alpha_3 = 0$, which is of the form

$Ax = 0$, with $A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{bmatrix}$

$$[A\,B] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & -6 & 6 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + \frac{1}{2}R_2 \\ R_4 \to R_4 + 6R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[AB] = 2 =$ Rank of $A <$ no. of unknowns

$\therefore$ infinite solution.

$$\alpha_1 + \frac{3}{2}\alpha_2 + \frac{1}{2}\alpha_3 = 0$$

$$\alpha_2 - \alpha_3 = 0$$

Let $\alpha_3 = a$, then $\alpha_2 = a$

$$\therefore \alpha_1 + \frac{3}{2}a + \frac{1}{2}a = 0$$

$$\Rightarrow \alpha_1 + 2a = 0 \Rightarrow \alpha_1 = -2a$$

when $a = 1$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\alpha_3 = 1$

$\therefore$ S is not linearly independent.

$\therefore$ S is not a basis of $\mathbb{R}^4$.

## Coordinate matrix

Let $B = \{v_1, v_2, \dots v_n\}$ be an ordered basis for a vector space V and let u be a vector in V such that $u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$. Then the scalars $c_1, c_2, \dots c_n$ are said to be the coordinates of u relative to the basis B. The coordinate matrix (or coordinate vector) of u relative to B is the column matrix in $R^n$ whose components are the coordinates of u written as a column and is represented as $[u]_B$.

ie; If $u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$, then $[u]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

1) Find the coordinate matrix of $u = (3, 2)$ relative to the standard basis of $R^2$.

Ans: $B = \{(1,0), (0,1)\}$

Let $(3,2) = c_1(1,0) + c_2(0,1)$

$= (c_1, 0) + (0, c_2)$

$= (c_1, c_2)$

$\Rightarrow c_1 = 3$ and $c_2 = 2$

$\therefore [u]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. Find the coordinate matrix of $u=(3,2)$ relative to the non standard basis $\{(1,0),(1,1)\}$ of $\mathbb{R}^2$.

Ans: $B=\{(1,0),(1,1)\}$

Let $(3,2) = c_1(1,0)+c_2(1,1)$

$= (c_1,0)+(c_2,c_2)$

$=(c_1+c_2,\ c_2)$

$\Rightarrow c_1+c_2=3$ and $c_2=2$

$\therefore c_1 = 3-c_2$

$= 3-2=1$

So $[u]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. Find the coordinate matrix of $x \in \mathbb{R}^2$ relative to the non standard basis $\{(4,0),(0,3)\}$.

~~Let~~ $B=\{(4,0),(0,3)\}$ and let $x=(x,y)$.

Let $(x,y) = c_1(4,0)+c_2(0,3)$

$= (4c_1,0)+(0,3c_2)$

$=(4c_1,3c_2)$

$\Rightarrow 4c_1 = x \Rightarrow c_1 = \frac{x}{4}$

$3c_2 = y \Rightarrow c_2 = \frac{y}{3}$

$\therefore [x]_B = \begin{bmatrix} \frac{x}{4} \\ \frac{y}{3} \end{bmatrix}$

4. Find the coordinate matrix of $X = (1, 2, -1)$ in $R^3$ relative to the non standard basis

$B' = \{(1,0,1), (0,-1,2), (2,3,-5)\}$

Ans: Let $(1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$

$= (c_1 + 2c_3, -c_2 + 3c_3, c_1 + 2c_2 - 5c_3)$

$\Rightarrow c_1 + 2c_3 = 1$

$-c_2 + 3c_3 = 2$

$c_1 + 2c_2 - 5c_3 = -1$ , which is of the form $Ax = B$

$\therefore [AB] = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{bmatrix}$ $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 2 & -7 & -2 \end{bmatrix}$ $R_2 \to \frac{R_2}{-1}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 2 & -7 & -2 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ $R_3 \to \frac{R_3}{-1}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & +1 & -2 \end{bmatrix}$

$\therefore$ Rank of $[AB]$ = Rank of $A = 3$ = no. of unknowns.

So unique solution.

$c_1 + 2c_3 = 1$

$c_2 - 3c_3 = -2$

$c_3 = -2$

$\therefore c_2 - 3c_3 = -2 \Rightarrow c_2 + 6 = -2$

$\Rightarrow c_2 = -2 - 6 = -8$

$c_1 + 2c_3 = 1 \Rightarrow c_1 - 4 = 1 \Rightarrow c_1 = 5$

$\therefore [x]_B = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$

5. Find the coordinate matrix of $u = (1, -1)$ relative to the non standard basis $\{(1,0), (1,1)\}$ in $R^2$.

Ans: Let $(1, -1) = \alpha_1 (1,0) + \alpha_2 (1,1)$

$= (\alpha_1 + \alpha_2, \alpha_2)$

$\Rightarrow \alpha_1 + \alpha_2 = 1$ and $\alpha_2 = -1$

$\therefore \alpha_1 - 1 = 1 \Rightarrow \alpha_1 = 2$

$\therefore$ Co-ordinate matrix, $[u]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

6. Find the coordinate matrix of $x \in R^3$ relative to the non standard ordered basis $B = \{(1,1,0), (0,1,1), (1,0,1)\}$. Using this result, find $[(1,1,1)]_B$.

Ans. $B = \{(1,1,0), (0,1,1), (1,0,1)\}$ and $x = (x, y, z)$.

Let $(x, y, z) = \alpha_1 (1,1,0) + \alpha_2 (0,1,1) + \alpha_3 (1,0,1)$

$= (\alpha_1 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3)$

$\Rightarrow \alpha_1 + \alpha_3 = x, \quad \alpha_1 + \alpha_2 = y, \quad \alpha_2 + \alpha_3 = z.$

which is of the form $Ax = B$.

$$\therefore [AB] = \begin{bmatrix} 1 & 0 & 1 & x \\ 1 & 1 & 0 & y \\ 0 & 1 & 1 & z \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 1 & 1 & z \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 0 & 2 & z-(y-x) \end{bmatrix} \quad R_3 \to \frac{R_3}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 0 & 1 & \frac{1}{2}(z-y+x) \end{bmatrix}$$

Rank of $[AB] = 3 =$ Rank of $A =$ no. of unknowns

$\therefore$ unique soln.

$\alpha_1 + \alpha_3 = x$

$\alpha_2 - \alpha_3 = y - x$

$\alpha_3 = \frac{1}{2}(x - y + z)$

$\therefore \alpha_2 - \alpha_3 = y - x \Rightarrow \alpha_2 - \frac{1}{2}(x-y+z) = y - x$

$\Rightarrow \alpha_2 = y - x + \frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}z$

$= -\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z$

$= \frac{1}{2}(-x + y + z)$

$\alpha_1 + \alpha_3 = x \Rightarrow \alpha_1 + \frac{1}{2}(x - y + z) = x$

$\Rightarrow \alpha_1 = x - \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z$

$= \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z = \frac{1}{2}(x+y-z)$

$$\therefore [X]_B = \begin{bmatrix} \frac{1}{2}(x+y-z) \\ \frac{1}{2}(-x+y+z) \\ \frac{1}{2}(x-y+z) \end{bmatrix}$$

$$\therefore [(1,1,1)]_B = \begin{bmatrix} \frac{1}{2}(1+1-1) \\ \frac{1}{2}(-1+1+1) \\ \frac{1}{2}(1-1+1) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

7. The coordinate matrix of $x \in \mathbb{R}^2$ relative to a non standard basis $B = \{(2,-1), (0,1)\}$ for $\mathbb{R}^2$ is given as $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Find the coordinate matrix of $x$ relative to the standard basis of $\mathbb{R}^2$.

Ans: $B = \{(2,-1), (0,1)\}$ and let $x = (x, y)$

Given $[x]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$\therefore (x,y) = 4(2,-1) + 1(0,1)$$
$$= (8,-4) + (0,1)$$
$$= (8,-3)$$

$\Rightarrow x = 8, \quad y = -3$

~~Let the~~ Standard basis $B' = \{(1,0), (0,1)\}$

$$(8,-3) = \alpha_1(1,0) + \alpha_2(0,1)$$
$$= (\alpha_1, \alpha_2)$$

$\Rightarrow \alpha_1 = 8, \quad \alpha_2 = -3$

$$\therefore [x]_{B'} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

8. The coordinate matrix of $x \in \mathbb{R}^3$ relative to a non standard ordered basis $B = \{(1,0,1), (1,1,0), (0,1,1)\}$ for $\mathbb{R}^3$ is given as $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Find the coordinate matrix of $x$ relative to the standard basis of $\mathbb{R}^3$.

Ans: Let $x = (x, y, z)$. Given $[x]_B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$$\therefore (x, y, z) = 2(1,0,1) + 3(1,1,0) + 1(0,1,1)$$

$$= (2+3,\ 3+1,\ 2+1)$$

$$= (5, 4, 3)$$

$\therefore x = 5, \quad y = 4, \quad z = 3$

Standard basis $B' = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$(5,4,3) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

$$= (c_1, c_2, c_3)$$

$\Rightarrow c_1 = 5, \quad c_2 = 4, \quad c_3 = 3$

$$\therefore [x]_{B'} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

## Transition Matrix

Let $B$ and $B'$ be two bases for a vector space $V$. A matrix $P$ is called transition matrix from $B'$ to $B$ if $P[x]_{B'} = [x]_B$ for all $x \in V$.

**Note**

1. If $P$ is the transition matrix from a basis $B'$ to a basis $B$ in $R^n$, then $P$ is invertible and the transition matrix from $B$ to $B'$ is $P^{-1}$.

2. The coordinate matrix relative to the basis $B'$ is given by $[x]_{B'} = P^{-1}[x]_B$

**Problems**

1. Find the transition matrix from $B$ to $B'$ for the bases of $R^3$ given by $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B' = \{(1,0,1), (0,-1,2), (2,3,-5)\}$

Ans: Consider matrix formed by the vectors of $B'$ and $B$ as columns, we get

$$[B' \; B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right]$$

Now reducing the matrix to the form $[I_n \; P^{-1}]$, we get

$$R_3 \to R_3 - R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & -7 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \to \frac{R_2}{-1} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 2 & -7 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 2R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array}\right]$

$R_3 \to \dfrac{R_3}{-1} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$

$R_1 \to R_1 - 2R_3$
$R_2 \to R_2 + 3R_3$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$

$\therefore$ The transition matrix from $B$ to $B'$ is given by $P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}$

2. Find the transition matrix from $B$ to $B'$ where $B = \{(1,3), (-2,-2)\}$ and $B' = \{(-12,0), (-4,4)\}$ are two bases of $\mathbb{R}^2$. Hence find the coordinate matrix $[x]_{B'}$ if $[x]_B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Ans: $[B' \ B] = \left[\begin{array}{cc|cc} -12 & -4 & 1 & -2 \\ 0 & 4 & 3 & -2 \end{array}\right]$

Reducing the matrix to the form $[I_n, P^{-1}]$ we get

$R_1 \to \dfrac{R_1}{-12} \sim \left[\begin{array}{cc|cc} 1 & 1/3 & -1/12 & 1/6 \\ 0 & 4 & 3 & -2 \end{array}\right]$

$R_2 \to \dfrac{R_2}{4} \sim \left[\begin{array}{cc|cc} 1 & 1/3 & -1/12 & 1/6 \\ 0 & 1 & 3/4 & -1/2 \end{array}\right]$

$R_1 \to R_1 - \frac{1}{3}R_2 \sim \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 3/4 & -1/2 \end{array}\right]$

$\therefore P^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 3/4 & -1/2 \end{bmatrix}$

$[x]_{B'} = P^{-1}[x]_B = \begin{bmatrix} -1/3 & 1/3 \\ 3/4 & -1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -9/4 \end{bmatrix}$

3. Find the transition matrix from B to B' wh

HW B and B' are two bases of $R^3$ given by

$B = \{(1,3,4), (2,-5,2), (-4,2,-6)\}$ and

$B' = \{(1,2,-2), (4,1,-4), (-2,5,8)\}$. Hence fi

the coordinate matrix $[x]_{B'}$ if $[x]_B = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

Ans: $[B' \; B] = \left[\begin{array}{ccc|ccc} 1 & 4 & -2 & 1 & 2 & -4 \\ 2 & 1 & 5 & 3 & -5 & 2 \\ -2 & -4 & 8 & 4 & 2 & -6 \end{array}\right]$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 2R_1$ $\sim \left[\begin{array}{ccc|ccc} 1 & 4 & -2 & 1 & 2 & -4 \\ 0 & -7 & 9 & 1 & -9 & 10 \\ 0 & 4 & 4 & 6 & 6 & -14 \end{array}\right]$

$R_2 \to \frac{R_2}{-7} \sim \left[\begin{array}{ccc|ccc} 1 & 4 & -2 & 1 & 2 & -4 \\ 0 & 1 & -9/7 & -1/7 & 9/7 & -10/7 \\ 0 & 4 & 4 & 6 & 6 & -14 \end{array}\right]$

$R_3 \to R_3 - 4R_2$, $R_1 \to R_1 - 4R_2$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & +22/7 & 11/7 & -22/7 & 12/7 \\ 0 & 1 & -9/7 & -1/7 & 9/7 & -10/7 \\ 0 & 0 & 64/7 & \frac{46}{7} & 6/7 & -58/7 \end{array}\right]$

$R_3 \to R_3 \times \frac{7}{64} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 22/7 & 11/7 & -22/7 & 12/7 \\ 0 & 1 & -9/7 & -1/7 & 9/7 & -10/7 \\ 0 & 0 & 1 & 46/64 & 6/64 & -58/64 \end{array}\right]$

$$R_1 \to R_1 - \frac{22}{7}R_3 \qquad R_2 \to R_2 + \frac{9}{7}R_3 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11/16 & -55/16 & 73/16 \\ 0 & 1 & 0 & 175/224 & 315/224 & -581/224 \\ 0 & 0 & 1 & 23/32 & 3/32 & -29/32 \end{array}\right]$$

$\therefore$ Transition matrix $P^{-1} = \begin{bmatrix} -11/16 & -55/16 & 73/16 \\ 175/224 & 315/224 & -581/224 \\ 23/32 & 3/32 & -29/32 \end{bmatrix}$

$$[x]_{B'} = P^{-1}[x]_B$$

$$= \begin{bmatrix} -11/16 & -55/16 & 73/16 \\ 175/224 & 315/224 & -581/224 \\ 23/32 & 3/32 & -29/32 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 157/16 \\ -191/32 \\ -81/32 \end{bmatrix}$$

4. Find the basis $B$ of $\mathbb{R}^3$ such that $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is the transition matrix from $B$ to the basis $B' = \{(1,1,1), (1,1,0), (1,0,0)\}$

Ans: Let $B = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$

Since $P$ is the transition matrix from $B$ to $B'$, we have

$$(x_1, y_1, z_1) = 1(1,1,1) + 0(1,1,0) + 0(1,0,0)$$
$$= (1,1,1)$$

$(x_2, y_2, z_2) = 0(1,1,1) + 3(1,1,0) + 1(1,0,0)$

$= (4,3,0)$

$(x_3, y_3, z_3) = 0(1,1,1) + 2(1,1,0) + 1(1,0,0)$

$= (3,2,0)$

$\therefore B = \{(1,1,1), (4,3,0), (3,2,0)\}$

5. Find the basis $B$ of $\mathbb{R}^3$ such that $P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \\ 2 & 1 & 0 \end{bmatrix}$

is the transition matrix from $B$ to the basis

$B' = \{(1,2,0), (0,3,2), (1,0,-1)\}$

Ans: Let $B = \{(x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3)\}$

Since $P$ is the transition matrix from $B$ to $B'$,

we have

$(x_1, y_1, z_1) = 1(1,2,0) + -1(0,3,2) + 2(1,0,-1)$

$= (3,-1,-4)$

$(x_2, y_2, z_2) = 0(1,2,0) + 3(0,3,2) + 1(1,0,-1)$

$= (1,9,5)$

$(x_3, y_3, z_3) = 2(1,2,0) - 2(0,3,2) + 0(1,0,-1)$

$= (2,-2,-4)$

$\therefore B = \{(3,-1,-4), (1,9,5), (2,-2,-4)\}$