MODULE - I
LINEAR ALGEBRA

Elementary row operations (transformations)

Any one of the following operations on a matrix is called an elementary transformation:

(i) interchange of two rows. Interchange of i^{th} and j^{th} rows is denoted as $R_i \leftrightarrow R_j$

(ii) Multiplication of a row by a non-zero number k.

Multiplication of each element of i^{th} row by a non-zero number k is denoted as $R_i \rightarrow kR_i$

(iii) Addition of k times the elements of a row to the corresponding elements of another row.

Addition of k times the elements of j^{th} row to the corresponding elements of i^{th} row are denoted as $R_i \rightarrow R_i + kR_j$

Note

If a matrix B is obtained from a matrix A by one or more elementary transformations, then B is said to be equivalent to A, and is denoted as $A \sim B$.

Row Echelon form and rank of a matrix.

A matrix is said to be an Echelon matrix if it satisfies the following conditions:

(i) The first non-zero number in any row is 1

(ii) ~~correct~~ The remaining elements below 1 are ~~is~~ 0's.

(iii) All rows consisting entirely of zeros appear at the bottom of the matrix.

Eg. (1) $A = \begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (2) $B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note

The no. of non-zero rows in the echelon form is the rank of the matrix. and rank of A is denoted as $\rho(A)$

Q. Find the rank of the following matrices by reducing to echelon form

1. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 2R_1$ $\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix}$

$R_2 \to \frac{R_2}{2}$ $\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 2 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$ $\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{bmatrix}$

$R_3 \to \frac{R_3}{-4}$ $\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

which is in echelon form and rank $= \underline{\underline{3}}$

2. $\begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

$R_1 \to \frac{R_1}{2} \sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

which is in echelon form and rank $= \underline{\underline{2}}$

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 8R_1$ $\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix}$

$R_2 \to \frac{R_2}{-3} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2/3 & 1 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$
$R_4 \to R_4 + 15R_2$ $\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is in echelon form and rank $= \underline{\underline{2}}$

4. Reduce the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ to Row echelon form and hence find its rank.

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

$R_2 \to \frac{R_2}{5} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

$R_3 \to R_3 - 4R_2$

$R_4 \to R_4 - 9R_2$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$

$R_3 \to R_3 \cdot \frac{5}{33} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & 22/33 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$

$R_4 \to R_4 - \frac{33}{5}R_3 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is in echelon form and rank $= \underline{\underline{3}}$

5. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ Rank = 4

6. $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ Rank = 2

Linear System of Equations

A linear system of m equations in n unknowns $x_1, x_2, \dots x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{---} (1)$$

The system is called linear because each variable x_j appears in the first power only.

If $b_1, b_2, \dots b_m$ are all zeros then eqn (1) is called a homogeneous system.

If atleast one b_j is not zero, then eqn (1) is called a nonhomogeneous system.

A solution of (1) is a set of numbers $x_1, x_2, \dots x_n$ that satisfies all the m equations.

If the system (1) is homogeneous, it always has atleast the trivial solution $x_1 = 0,\ x_2 = 0, \dots x_n = 0$.

Matrix representation of equation (1) is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

ie, $AX = B$, where

A is the co-efficient matrix

X is the column matrix of unknowns

B is the column matrix of constants.

Augmented matrix

Two matrices A and B writing side by side is called Augmented matrix and is denoted as $[A : B]$

ie; $$[A\ B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \quad (1)$$

The system of equation (1) is consistent if it has atleast one solution & and inconsistent if it has no solution.

Gauss Elimination and Back substitution.

It is a method for solving linear system of equations. This process involves ~~determ~~ elimination and back substitution. For this we first form the augmented matrix $[A : B]$ for the system $AX = B$. By converting the augmented matrix $[A : B]$ to echelon form, we have to find the rank of $[A : B]$ and the rank of A.

Fundamental Theorem

1) If rank of $A \neq$ rank of $[A\ B]$, then the system is inconsistent. ie; the system has no solution.

2) If rank of A = rank of $[A B]$ = no. of unknowns, the system is consistent and has unique solution.

3) If rank of A = rank of $[A B] <$ no. of unknowns the system is consistent and has infinite solution obtained by giving $(n-r)$ variable an arbitrary value and using backward substitution (n - no. of variables, r - rank)

1) Solve the system of equations by Gauss Elimination method.

$3x + 3y + 2z = 1, \quad x + 2y = 4, \quad 10y + 3z = -2, \quad 2x - 3y - z = 5$

$AX = B \Rightarrow \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$

$[A\ B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$
$R_4 \to R_4 - 2R_1$
$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$

$R_2 \to \dfrac{R_2}{-3} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$

$R_3 \to R_3 - 10R_2$
$R_4 \to R_4 + 7R_2$
$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$

$R_3 \to R_3 \times \dfrac{3}{29} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -116/29 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$

$R_4 \to R_4 + \dfrac{17}{3}R_3 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -116/29 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank of $A = 3 =$ rank of $[AB] =$ no. of unknowns

∴ Given system is consistent and has unique solution

From echelon form $x + 2y = 4$

$y - \frac{2}{3}z = \frac{11}{3}$

$z = \frac{-116}{29} = -4$

∴ $y = \frac{11}{3} + \frac{2z}{3} = \frac{11}{3} - \frac{8}{3} = \frac{3}{3} = 1$

$x = 4 - 2y = 4 - 2 = 2$

∴ $x = 2, \quad y = 1, \quad z = -4$

2. $x + y + z = 6, \quad x + 2y - 3z = -4, \quad -x - 4y + 9z = 18$

$$[A\,B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -3 & -4 \\ -1 & -4 & 9 & 18 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -3 & 10 & 24 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

$R_3 \to \frac{R_3}{-2}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

rank of A = 3 = rank of [A B] = no. of unknowns

∴ System is consistent and has unique solution

By back substitution $x + y + z = 6$

$y - 4z = -10$

$z = 3$

$y = -10 + 12 = 2$

$x = 6 - 2 - 3 = 6 - 5 = 1$

∴ $x = 1, \quad y = 2, \quad z = 3$

3. $x+2y-z=3,\quad 3x-y+2z=1,\quad 2x-2y+3z=2,\quad x-y+z=-1$

$$[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_2 \to \dfrac{R_2}{-7}$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2$
$R_4 \to R_4 + 3R_2$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix}$$

$R_3 \to R_3 \cdot \dfrac{7}{5}$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix}$$

$R_4 \to R_4 + \dfrac{1}{7}R_3$
$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ \rho(A) = \rho(AB) = 3 =$ no. of unknowns

$\therefore$ system is consistent and has unique solution

By back substitution $\quad x+2y-z=3$

$$y - \frac{5}{7}z = \frac{8}{7}$$

$$z = 4$$

$$y = \frac{8}{7} + \frac{20}{7} = 4$$

$$x = 3 - 2y + z = 3 - 8 + 4$$
$$= 3 - 4 = \underline{\underline{-1}}$$

4. $x_1 - x_2 + x_3 = 0$, $-x_1 + x_2 - x_3 = 0$, $10x_2 + 25x_3 = 90$

$20x_1 + 10x_2 = 80$

$$[AB] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 20R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

$$R_2 \Leftrightarrow R_4 \quad \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{30} \quad \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/3 & 8/3 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - 10R_2 \quad \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/3 & 8/3 \\ 0 & 0 & 95/3 & 190/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 \cdot \frac{3}{95} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/3 & 8/3 \\ 0 & 0 & 1 & 190/95 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3 =$ no. of unknowns

$\therefore$ System is consistent and has unique solution.

$x - y + z = 0$

$y - \frac{2}{3}z = 8/3$

$z = \frac{190}{95} = 2$

$\therefore y = \frac{8}{3} + \frac{4}{3} = \frac{12}{3} = 4$

$x = y - z = 4 - 2 = 2$

$\therefore x_1 = 2, \quad x_2 = 4, \quad x_3 = 2$

5. $3x+2y+z=3, \quad 2x+y+z=0, \quad 6x+2y+4z=6$

$$[AB] = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{3} \quad \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

$$R_2 \to R_2 \cdot -3 \quad \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2 \quad \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$\therefore \rho(A) = 2$ and $\rho(AB) = 3$

$\therefore \rho(A) \neq \rho(AB)$

So the system is inconsistent and has no solution.

6. Solve $4x+y=4, \quad 5x-3y+z=2, \quad -9x+2y-z=5$

Ans: inconsistent and has no solution.

7. Solve $-2y-2z=8, \quad 3x+4y-5z=8$

$$[AB] = \begin{bmatrix} 0 & -2 & -2 & 8 \\ 3 & 4 & -5 & 8 \end{bmatrix}$$

$$R \to \frac{R_1}{-2} \quad \sim \begin{bmatrix} 0 & 1 & 1 & -4 \\ 3 & 4 & -5 & 8 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1 \sim \begin{bmatrix} 0 & 1 & 1 & -4 \\ 3 & 0 & -9 & 24 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{3} \sim \begin{bmatrix} 0 & 1 & 1 & -4 \\ 1 & 0 & -3 & 8 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 2 < 3$ the no. of unknowns

$\therefore$ system is consistent and has infinite solution.

From echelon form $y + z = -4$

$x - 3z = 8$

$n = 3, \ r = 2$

$\therefore 3 - 2 = 1$ variable should assign an arbitrary value

put $z = t$

$\therefore y = -4 - t$

$x = 8 + 3t$

8. $y + z - 2w = 0, \quad 2x - 3y - 3z + 6w = 2, \quad 4x + y + z - 2w = 4$

$$[AB] = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{2} \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2 \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$, $\rho(AB) = 2$ < no. of unknowns

$\therefore$ system is consistent and has infinite no. of solutions.

$n = 4$ $r = 2$

$\therefore n - r = 4 - 2$ variables should assign arbitrary values.

From echelon form $y + z - 2w = 0$

$x = 1$

Put $w = a$ $z = b$

$\therefore y = 2w - z = 2a - b$

9. Show that the system of equations are inconsistent

$2x + 6y = -11$, $6x + 20y - 6z = -3$, $6y - 18z = -1$.

10. Find the values of μ for which the system of equations $x + y + z = 1$, $x + 2y + 3z = \mu$, and $x + 5y + 9z = \mu^2$ will be consistent. For each value of μ obtained, find the solution of the system

$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu - 1 \\ 0 & 4 & 8 & \mu^2 - 1 \end{bmatrix}$

$R_3 \to R_3 - 4R_2 \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & \mu^2-4\mu+3 \end{bmatrix}$

System will be consistent if $\rho(A) = \rho(AB)$

Here $\rho(A) = 2$

$\therefore$ $\rho(AB)$ should also be 2. This is possible only if $\mu^2 - 4\mu + 3 = 0$ ~~and $\mu \neq 0$~~

ie; $\mu^2 - 4\mu + 3 = 0$

$\Rightarrow (\mu-3)(\mu-1) = 0$

$\Rightarrow \mu = 3, 1$

when $\mu = 3$ $[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore \rho(A) = \rho(AB) = 2 <$ no. of unknowns

$\therefore$ System has infinite no. of solutions

$n = 3$, $r = 2$ $\therefore$ $3 - 2 = 1$ variable should assign arbitrary value.

we have $x + y + z = 1$

$y + 2z = 2$

put $z = t$ $\therefore$ $y = 2 - 2t$

$x = 1 - y - z = 1 - (2 - 2t) - t$

$= -1 + t = t - 1$

when $\mu = 1$ $[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore \rho(A) = \rho(AB) = 2 <$ no. of unknowns

$\therefore$ system has infinite no. of solutions

$n = 3, r = 2 \quad \therefore n - r = 3 - 2 = 1$ variable should assign arbitrary value.

we have $x + y + z = 1$

$y + 2z = 0$

$z = t \Rightarrow y = -2t$

$x = 1 - y - z = 1 + 2t - t = \underline{1 + t}$

11. Find the values of a and b for which the system of equations $x + y + 2z = 2$, $2x - y + 3z = 10$, $5x - y + az = b$ has (i) no solution (2) unique soln (3) infinite no. of solutions.

$$[A\ B] = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 10 \\ 5 & -1 & a & b \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & 6 \\ 0 & -6 & a-10 & b-10 \end{bmatrix}$$

$R_2 \rightarrow \dfrac{R_2}{-3}$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1/3 & -2 \\ 0 & -6 & a-10 & b-10 \end{bmatrix}$$

$a - 10 + 2$

$b - 10 - 12$

$R_3 \rightarrow R_3 + 6R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1/3 & 2 \\ 0 & 0 & a-8 & b-22 \end{bmatrix}$$

The system has no solution when $\rho(A) \neq \rho(AB)$

$\therefore$ $a-8=0$ and $b-22 \neq 0$ for no solution.

ie; $a=8$ and $b \neq 22$.

The system has unique solution when $\rho(A)=\rho(AB)$ = no. of unknowns

$\therefore$ $\rho(A)=\rho(AB)=3$

$\therefore$ $a-8 \neq 0$ and $b-22$ can be any value

$\therefore$ $a \neq 8$ and b can be any value

The system has infinite solution when $\rho(A)=\rho(AB)$ < no. of unknowns

$a-8=0$ and $b-22=0$

ie; $\underline{a=8}$ and $\underline{b=22}$

11 Solve $x+2y+3z=0$, $3x+4y+4z=0$, $7x+10y+12z=0$

$$[AB] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$z=0$, $y=0$, $x=0$

12 $x_1+3x_2+2x_3=0$, $2x_1-x_2+3x_3=0$, $3x_1-5x_2+4x_3=0$

$x_1+17x_2+4x_3=0$

$x_1 = -\frac{11t}{7}$, $x_2 = \frac{-t}{7}$, $x_3=t$

13 $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+az=b$

no soln $a=8$ $b \neq 15$

Eigen values and Eigen vectors.

The problem of determining the eigen values and eigen vectors of a matrix is called an eigen value problem. Let A be a square matrix of order n, I corresponding identity matrix, λ a scalar, then matrix $[A-\lambda I]$ is called the characteristic matrix. The determinant $|A-\lambda I|$ is called the characteristic determinant of A. By developing $|A-\lambda I|$ we obtain a polynomial of n^{th} degree in λ and this is called characteristic polynomial of A.

$|A-\lambda I|=0$ is called the charactoristic equation of A and its roots are called eigen value or characteristic roots or latent roots of A.

The set of all eigen value of A is called the spectrum of A. The largest of the absolute value of eigen value of A is called the spectral radius of A.

Let the roots of the characteristic eqn be $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$. Corresponding to each of these eigen values the system of equation $[A-\lambda I]X=0$ have a non-trivial solution. These solutions are known as eigen vectors corresponding to $\lambda_1, \lambda_2, \dots \lambda_n$

1. Find the eigen values and the corresponding eigen vectors for the following matrices.

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

$|A - \lambda I| = (-5-\lambda)(-2-\lambda) - 4$

$= 10 + 7\lambda + \lambda^2 - 4$

$= \lambda^2 + 7\lambda + 6$

$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow (\lambda + 6)(\lambda + 1) = 0$

$\Rightarrow \lambda = -6, -1$

$\therefore$ Eigen values are $\lambda = \underline{\underline{-6, -1}}$

~~when $\lambda = -6$~~

$[A - \lambda I] X = 0$

$$\Rightarrow \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} (-5-\lambda)x_1 + 2x_2 &= 0 \\ 2x_1 + (-2-\lambda)x_2 &= 0 \end{aligned}\right\} \text{ --- } ①$$

when $\lambda = -6$

① $\Rightarrow$ $x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$[A\ B] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Rank $= 1$

$\therefore$ Considering the first equation $x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$

$\therefore \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -6$

when $\lambda = -1$

$① \Rightarrow -4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$

$[AB] = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

Rank $= 1$

considering the first equation $-4x_1 + 2x_2 = 0$

$\Rightarrow -4x_1 = -2x_2$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{-4}$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2}$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$

Note

1) If given matrix A is a 2×2 matrix, i.e; if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$|A - \lambda I| = \lambda^2 - (\text{sum of diagonal elts of } A)\lambda + |A|$

$= \lambda^2 - (a_{11} + a_{22})\lambda + |A|$

2. If A is a 3×3 matrix, then ie $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A-\lambda I| = \lambda^3 - (\text{sum of diagonal elts of } A)\lambda^2 + (\text{sum of } \cancel{\text{co-factors}} \text{ minors of diagonal elts of } A)\lambda - |A|.$

$= \lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + (M_{11}+M_{22}+M_{33})\lambda - |A|$

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

2. $A = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \lambda^2 - 13\lambda + 30 = 0$

$\Rightarrow (\lambda-10)(\lambda-3) = 0$

$\Rightarrow \lambda = 10, 3$

$[A-\lambda I]X = 0$

$\Rightarrow \begin{bmatrix} 6-\lambda & 3 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— (1)

$\Rightarrow \left.\begin{aligned} (6-\lambda)x_1 + 3x_2 &= 0 \\ 4x_1 + (7-\lambda)x_2 &= 0 \end{aligned}\right\}$ —— (1)

when $\lambda = 10$

(1) $\Rightarrow -4x_1 + 3x_2 = 0$

$4x_1 - 3x_2 = 0$

$[A\,B] = \begin{bmatrix} -4 & 3 & 0 \\ 4 & -3 & 0 \end{bmatrix}$

Rank $= 1$

considering the first equation $-4x_1 + 3x_2 = 0$

$\Rightarrow -4x_1 = -3x_2$

$\Rightarrow \dfrac{x_1}{-3} = \dfrac{x_2}{-4}$

$\Rightarrow \dfrac{x_1}{3} = \dfrac{x_2}{4}$

$\therefore \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 10$

when $\lambda = 3$

(1) $\Rightarrow 3x_1 + 3x_2 = 0$

$4x_1 + 4x_2 = 0$

Considering the first equation $3x_1 + 3x_2 = 0$

$$\Rightarrow 3x_1 = -3x_2$$

$$\Rightarrow \frac{x_1}{-3} = \frac{x_2}{3}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1}$$

$\therefore \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

3. $A = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}$

$|A - \lambda I| = 0 \quad \Rightarrow \lambda^2 - \left(\frac{3}{2} + 3\right)\lambda + \frac{3}{2} \cdot 3 = 0$

$$\Rightarrow \lambda^2 - \frac{9}{2}\lambda + \frac{9}{2} = 0$$

$$\Rightarrow \lambda = \frac{3}{2}, 3$$

$[A - \lambda I]x = 0$

$$\Rightarrow \begin{bmatrix} \frac{3}{2} - \lambda & 0 \\ 0 & 3 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

when $\lambda = 3/2$

~~$\left(\frac{3}{2} - \lambda\right)x_1 = 0$~~

~~$(3 - \lambda)x_2 = 0$~~

(1) $\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \frac{3}{2} x_2 = 0$$

$$\Rightarrow 0x_1 + \frac{3}{2} x_2 = 0$$

$$\Rightarrow 0x_1 = \frac{-3}{2} x_2$$

$$\Rightarrow 0x_1 = -3x_2$$

$$\Rightarrow \frac{x_1}{-3} = \frac{x_2}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0}$$

$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3/2$

when $\lambda = 3$

$$① \Rightarrow \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{3}{2} x_1 + 0 x_2 = 0$$

$$\Rightarrow \frac{3}{2} x_1 = -0 x_2$$

$$\Rightarrow 3x_1 = 0 x_2$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{3} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1}$$

$\therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

4. $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \lambda^2 - 0\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda^2 = -16 \Rightarrow \lambda = \pm 4i$$

$$[A - \lambda I] x = 0 \Rightarrow \begin{bmatrix} -\lambda & 4 \\ -4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ --- } ①$$

when $\lambda = 4i$

$$① \Rightarrow \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4i x_1 + 4x_2 = 0$$

$$-4x_1 - 4i x_2 = 0$$

$$\therefore -4i x_1 = -4x_2$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-4i} \Rightarrow \frac{x_1}{1} = \frac{x_2}{i}$$

$\therefore \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4i$

when $\lambda = -4i$

$$① \Rightarrow \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4ix_1 + 4x_2 = 0$

$-4x_1 + 4ix_2 = 0$

$4ix_1 + 4x_2 = 0 \Rightarrow x_1 i = -x_2$

$\Rightarrow \dfrac{x_1}{-1} = \dfrac{x_2}{i}$

$\therefore \begin{bmatrix} -1 \\ i \end{bmatrix}$ is the eigen vector.

5. $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 12\lambda^2 + (15+8+16)\lambda - 28 = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$

$\Rightarrow \lambda = 1, 7, 4$

① $[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to 0$

when $\lambda = 1$

① $\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 + 4x_2 = 0$

$-2x_1 + 2x_3 = 0$

Rank = 2 consider I^{st} and II^{nd} equations,

$\begin{matrix} 2 & -2 & 3 & 2 \\ 4 & 0 & 2 & 4 \end{matrix}$

$\dfrac{x_1}{0-(-8)} = \dfrac{x_2}{-4-0} = \dfrac{x_3}{12-4} \Rightarrow \dfrac{x_1}{8} = \dfrac{x_2}{-4} = \dfrac{x_3}{8}$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

when $\lambda = 7$

$$① \Rightarrow \begin{bmatrix} -3 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 - 4x_3 = 0$$

Rank = 2, considering I^{st} and II^{nd} equations,

$$\begin{matrix} 2 & -2 & -3 & 2 \\ -2 & 0 & 2 & -2 \end{matrix}$$

$$\frac{x_1}{0-(4)} = \frac{x_2}{-4-0} = \frac{x_3}{6-4}$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-4} = \frac{x_3}{2} \Longrightarrow \frac{x_1}{-2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 7$

when $\lambda = 4$

$$① \Rightarrow \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 0x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 + x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 - x_3 = 0$$

2 -2 0 2

1 0 2 1

$$\frac{x_1}{0-(-2)} = \frac{x_2}{-4-0} = \frac{x_3}{0-4}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{-4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{-2}$$

$\therefore \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4$

HW 6. $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

$\Rightarrow \lambda = 1, 4, 3$

$$[A-\lambda I]X = 0 \Rightarrow \begin{bmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

when $\lambda = 1$

$$(1) \Rightarrow \begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 + 5x_2 + 3x_3 = 0$

$0x_1 + 3x_2 + 6x_3 = 0$

5 3 2 5

3 6 0 3

$$\frac{x_1}{30-9} = \frac{x_2}{0-12} = \frac{x_3}{6-0}$$

$$\frac{x_1}{7} = \frac{x_2}{-4} = \frac{x_3}{2} \quad \therefore \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$$

when $\lambda = 4$

$$(1) \Rightarrow \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \; -x_1 + 5x_2 + 3x_3 = 0$$
$$0x_1 + 0x_2 + 6x_3 = 0$$
$$0x_1 + 0x_2 - 3x_3 = 0$$

$$\begin{matrix} 5 & 3 & -1 & 5 \\ 0 & 6 & 0 & 0 \end{matrix}$$

$$\frac{x_1}{30} = \frac{x_2}{6} = \frac{x_3}{0} \Rightarrow \frac{x_1}{5} = \frac{x_2}{1} = \frac{x_3}{0} \quad \therefore \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

when $\lambda = 3$

$$(1) \Rightarrow \begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + 5x_2 + 3x_3 = 0$$
$$0x_1 + x_2 + 6x_3 = 0$$
$$0x_1 + 0x_2 - 2x_3 = 0$$

$$\begin{matrix} 5 & 3 & 0 & 5 \\ 1 & 6 & 0 & 1 \end{matrix}$$

$$\frac{x_1}{27} = \frac{x_2}{0} = \frac{x_3}{0} \quad \therefore \begin{bmatrix} 27 \\ 0 \\ 0 \end{bmatrix}$$

6. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$
$$\Rightarrow \lambda = 5, -3, -3$$

$$[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

When $\lambda = 5$

$$① \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 - 5x_3 = 0$$

From I^{st} 2 eqns,

$$\begin{matrix} 2 & -3 & -7 & 2 \\ -4 & -6 & 2 & -4 \end{matrix}$$

$$\therefore \frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\Rightarrow \frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the eigen vector

When $\lambda = -3$

$$① \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 + 3x_3 = 0$$

Rank $= 1$ $\quad n - r = 3 - 1 = 2$ variables have to assign arbitrary value

$x_1 + 2x_2 - 3x_3 = 0$

Put $x_3 = t_1$, $x_2 = t_2$

$\therefore x_1 = 3t_1 - 2t_2$

Put $t_1 = 0$, $t_2 = 1$

$\therefore x_1 = -2$, $x_2 = 1$, $x_3 = 0$

Put $t_1 = 1$, $t_2 = 0$

$x_1 = 3$, $x_2 = 0$, $x_3 = 1$

$\therefore \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors

8. $\begin{bmatrix} 6 & 5 & 2 \\ 2 & 0 & -8 \\ 5 & 4 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\Rightarrow \lambda = 2, 2, 2$

$[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 6-\lambda & 5 & 2 \\ 2 & -\lambda & -8 \\ 5 & 4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (1)

when $\lambda = 2$ (1) $\Rightarrow \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow 4x_1 + 5x_2 + 2x_3 = 0$$

$$2x_1 - 2x_2 - 8x_3 = 0 \quad , \text{ rank} = 2$$

$$5x_1 + 4x_2 - 2x_3 = 0$$

$$\begin{array}{cccc} 5 & 2 & 4 & 5 \\ -2 & -8 & 2 & -2 \end{array}$$

$$\frac{x_1}{-40+4} = \frac{x_2}{4+32} = \frac{x_3}{-8-10}$$

$$\Rightarrow \frac{x_1}{-36} = \frac{x_2}{36} = \frac{x_3}{-18}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

Properties of Eigen values

1) The sum of eigen values of a matrix is the sum of diagonal elts of the matrix

2) The eigen values of a square matrix A and its transpose A^T are the same.

3) The product of the eigen values of A is equal to $|A|$.

4) If λ is an eigen value of a non singular square matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

5) If ~~λ_1, λ_2~~, λ is an eigen value of a nonsingulare square matrix A then $\frac{|A|}{\lambda}$ is an eigen value of adj A.

6 If λ is an eigen value of A, then $A-KI$ has the eigen value $\lambda - k$

7) If λ is an eigen value of A, then kA has the eigen value $k\lambda$.

8) If λ is an eigen of A, the A^m has the eigen value λ^m.

Problem

1. If 2 is an eigen value of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, without using ch-eqn, find the other eigen values

of the matrix A. Also find the eigen values of A^3, A^T, A^{-1}, $5A$, $A-3I$ and adj A

Ans: Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A

given $\lambda_1 = 2$

$\lambda_1 + \lambda_2 + \lambda_3$ = sum of diagonal elts $= 11$

$\Rightarrow 2 + \lambda_2 + \lambda_3 = 11$

$\Rightarrow \lambda_2 + \lambda_3 = 9$ — ①

Also $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$

$\Rightarrow 2\lambda_2\lambda_3 = 36$

$\Rightarrow \lambda_2\lambda_3 = 18$

$\Rightarrow \lambda_2 = \frac{18}{\lambda_3}$

$\therefore$ ① $\Rightarrow \frac{18}{\lambda_3} + \lambda_3 = 9$

$\Rightarrow \lambda_3^2 + 18 = 9\lambda_3$

$\Rightarrow \lambda_3^2 - 9\lambda_3 + 18 = 0$

$\Rightarrow \lambda_3 = 6, 3$

$\therefore \lambda_2 = \frac{18}{6}, \frac{18}{3} = 3, 6$

$\therefore \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$

Eigen values of A^3 are $2^3, 3^3, 6^3$

i.e; $8, 27, 216$

∴ Eigen values of A^T = eigen values of A

$= 2, 3, 6$

Eigen values of $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

Eigen values of $5A = \cancel{10, 15, 30}\ \ 5\lambda_1, 5\lambda_2, 5\lambda_3$

$= 10, 15, 30$

Eigen values of $A - 3I = \lambda_1 - 3, \lambda_2 - 3, \lambda_3 - 3$

$= -1, 0, 3$

Eigen values of adj $A = \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$

$= \frac{36}{2}, \frac{36}{3}, \frac{36}{6}$

$= \underline{\underline{18, 12, 6}}$

Diagonalisation of a matrix.

Let A be an $n\times n$ square matrix. If A has a basis of eigen vectors, then the matrix $D = X^{-1}AX$ is a diagonal matrix with the eigen values of A as the entries on the main diagonal and X is the matrix with the basis of eigen vectors as columns.

Steps of diagonalisation process

Let A be a square matrix of order 3.

1) Find the eigen values of A. Let it be $\lambda_1, \lambda_2, \lambda_3$

2) Find the eigen vectors of A. They must be independent. Let it be X_1, X_2, X_3.

3) Form the matrix $X = [X_1 \quad X_2 \quad X_3]$

4) Find $D = X^{-1}AX = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Note

If $D = X^{-1}AX$, then $D^m = X^{-1}A^m X$

$\Rightarrow A^m = XD^mX^{-1}$

1) Diagonalise $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$

$\Rightarrow \lambda = 2, 2, 5$

$[A-\lambda I]X = 0 \Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$

When $\lambda = 2$

① $\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$x_1 - x_2 + x_3 = 0$

when $x_3 = t_1 \quad x_2 = t_2 \quad x_1 = t_2 - t_1$

$t_1 = 0$
$t_2 = 1 \quad \Rightarrow \quad x_1 = 1, x_2 = 1, x_3 = 0$

$t_1 = 1$
$t_2 = 0 \quad \Rightarrow \quad x_1 = -1, x_2 = 0, x_3 = 1$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \& \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors

When $\lambda = 5$

① $\Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 - x_2 + x_3 = 0$

$-x_1 - 2x_2 - x_3 = 0$

$x_1 - x_2 - 2x_3 = 0$

Rank $= 2 \quad \therefore \quad -2x_1 - x_2 + x_3 = 0$

$-x_1 - 2x_2 - x_3 = 0$

$$\Rightarrow \begin{matrix} -1 & 1 & -2 & -1 \\ -2 & -1 & -1 & -2 \end{matrix}$$

$$\frac{x_1}{1+2} = \frac{x_2}{-1-2} = \frac{x_3}{4-1} \Rightarrow \frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector

$$\therefore X = \begin{array}{c} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \text{II} & \text{III} & \text{I} \end{array} \end{array}$$

$$\therefore D = X^{-1}AX = \begin{array}{c} \text{II} \\ \text{III} \\ \text{I} \end{array} \begin{bmatrix} 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

2. Diagonalise the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$ and hence find $A^{?}$

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 0\lambda^2 - 9\lambda + 0 = 0$

$\Rightarrow \lambda = 0, 3, -3$

$[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & -\lambda & 2 \\ 0 & 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ —— (1)

when $\lambda = 0$

(1) $\Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 + 0x_2 + 2x_3 = 0$

$0x_1 + 2x_2 - x_3 = 0$

From I^{st}, 2 eqns

$$\begin{array}{cccc} -2 & 0 & 1 & -2 \\ 0 & 2 & -2 & 0 \end{array}$$

$\dfrac{x_1}{-4} = \dfrac{x_2}{-2} = \dfrac{x_3}{-4} \rightarrow \dfrac{x_1}{2} = \dfrac{x_2}{1} = \dfrac{x_3}{2}$

$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector.

when $\lambda = 3$

(1) $\Rightarrow \begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 - 3x_2 + 2x_3 = 0$

$0x_1 + 2x_2 - 4x_3 = 0$

$\begin{matrix} -2 & 0 & -2 & -2 \\ -3 & 2 & -2 & -3 \end{matrix}$

$\frac{x_1}{-4} = \frac{x_2}{4} = \frac{x_3}{6-4} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{1}$

$\therefore \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ is the eigen vector

when $\lambda = -3$

(1) $\Rightarrow \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$4x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 + 3x_2 + 2x_3 = 0$

$\begin{matrix} -2 & 0 & 4 & -2 \\ 3 & 2 & -2 & 3 \end{matrix}$

$\frac{x_1}{-4} = \frac{x_2}{-8} = \frac{x_3}{8} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-2}$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector

$\therefore X = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

$$D = X^{-1} A X = \begin{bmatrix} 2/9 & 1/9 & 2/9 \\ -2/9 & 2/9 & 1/9 \\ 1/9 & 2/9 & -2/9 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$A^2 = X D^2 X^{-1} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}$$

3. Diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and hence find A^4.

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3$

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— ①}$$

when $\lambda = 1$

$$\text{①} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 0x_2 + x_3 = 0$

$0x_1 + x_2 + 0x_3 = 0$

$$\begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector

when $\lambda = 2$

(1) $\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 + 0x_2 + x_3 = 0$

$x_1 + 0x_2 + 0x_3 = 0$

$$\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$$

$\dfrac{x_1}{0} = \dfrac{x_2}{1} = \dfrac{x_3}{0}$ $\quad \therefore \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector

when $\lambda = 3$

(1) $\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1 + 0x_2 + x_3 = 0$

$0x_1 - x_2 + 0x_3 = 0$

$$\begin{array}{cccc} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{array}$$

$\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$ $\quad \therefore \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector

$\therefore X = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$D = X^{-1} A X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A^4 = X D^4 X^{-1} = \begin{bmatrix} 41 & 0 & 40 \\ 0 & 16 & 0 \\ 40 & 0 & 41 \end{bmatrix}$