

## Module-6

### Prandtl's Method for Torsional prismatic Bars

#### Assumption

- ① The material should be homogeneous and isotropic
- ② Angle of twist per unit length is a constant
- ③ Z component of displacement,  $w=0$
- ④ Magnitude of torque is same over entire length of bar

#### Step① Displacement Function

$$u = -yz\theta \quad w = \psi \psi(x, y)$$

$$v = zx\theta$$

#### Step② Strain components

$$\epsilon_{ij} = \begin{bmatrix} 0 & 0 & \theta \left( \frac{\partial \psi}{\partial z} - y \right) \\ 0 & 0 & \theta \left( \frac{\partial \psi}{\partial y} + x \right) \\ \theta \left( \frac{\partial \psi}{\partial x} - y \right) & \theta \left( \frac{\partial \psi}{\partial y} + x \right) & 0 \end{bmatrix}$$

#### Stress components

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & G\theta \left( \frac{\partial \psi}{\partial z} - y \right) \\ 0 & 0 & G\theta \left( \frac{\partial \psi}{\partial y} + x \right) \\ G\theta \left( \frac{\partial \psi}{\partial x} - y \right) & G\theta \left( \frac{\partial \psi}{\partial y} + x \right) & 0 \end{bmatrix}$$

#### Step 3 : Equilibrium Equations

$$\frac{\partial \sigma_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \Rightarrow 0 + 0 + \frac{\partial \tau_{xz}}{\partial z} = 0 \text{ (satisfied)}$$

$$\frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \Rightarrow 0 + 0 + \frac{\partial \tau_{yz}}{\partial z} = 0 \text{ (satisfied)}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0}$$

In order to satisfy 3rd condition we assume  $\phi(x, y)$  stress function

$$\boxed{\tau_{xz} = \frac{\partial \phi}{\partial y} \quad \& \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}}$$

where  $\phi(x, y)$  is a scalar function called Prandtl's Stress Function

#### Step 4 Governing Equation using compatibility Condition

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial z} \left( \frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial z} (-\tau_{yz}) = -\frac{\partial}{\partial z} \left[ G\theta \left( x + \frac{\partial \psi}{\partial y} \right) \right] = -G\theta \left[ \frac{\partial z}{\partial x} + \frac{\partial^2 \psi}{\partial z \partial y} \right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = -G\theta - G\theta \frac{\partial^2 \psi}{\partial z \partial y} \quad \text{--- (1)}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial z} \left( \frac{\partial \phi}{\partial y} \right) = \frac{\partial}{\partial z} \tau_{xz} = \frac{\partial}{\partial y} \left[ G\theta \left( \frac{\partial \psi}{\partial x} - y \right) \right]$$

$$= G\theta \frac{\partial^2 \psi}{\partial z \partial y} - G\theta \frac{\partial \psi}{\partial z} \Rightarrow \frac{\partial^2 \phi}{\partial y^2} = G\theta \frac{\partial^2 \psi}{\partial z \partial y} - G\theta \quad \text{--- (2)}$$

Equating L.H.S & R.H.S. OR eq(1) & (2)

$$\left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\sigma \right] \text{ OR } \boxed{\nabla^2 \phi = -2G\sigma}$$

{ This is governing equation of Torsion  
of prismatic bars by prandtl. It also called Poisson Equation.

Step 5: Boundary condition of Stress Function

Here by applying direction cosines of prandtl stress function, we get  
 $\phi$  between points is a constant.  $\boxed{d\phi=0}$

Step 6: Inque

$$T = \iint (\tau_{xy} - \tau_{yx}) dx dy = \iint x \left( -\frac{\partial \phi}{\partial x} \right) - y \left( \frac{\partial \phi}{\partial y} \right) dx dy \\ = - \left[ \iint x \frac{\partial \phi}{\partial x} dx dy + \iint y \frac{\partial \phi}{\partial y} dx dy \right]$$

Apply integration by parts  $\boxed{T = -2 \iint \phi dx dy}$

Shear Stress

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = G\sigma \left( \frac{\partial \psi}{\partial y} + x \right) \Rightarrow \frac{\partial \psi}{\partial y} = -\left( \frac{1}{G\sigma} \frac{\partial \phi}{\partial x} + x \right)$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = G\sigma \left( \frac{\partial \psi}{\partial x} - y \right) \Rightarrow \frac{\partial \psi}{\partial x} = \left( \frac{1}{G\sigma} \frac{\partial \phi}{\partial y} + y \right)$$

J-Integration in prandtl's Function

$$J = \iint \left( (x^2 + y^2) + x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) dx dy = \iint (x^2 + y^2) - x \left( \frac{1}{G\sigma} \frac{\partial \phi}{\partial x} + x \right) - y \left( \frac{1}{G\sigma} \frac{\partial \phi}{\partial y} + y \right) dx dy \\ \boxed{J = \frac{-1}{G\sigma} \left[ \iint x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right] dx dy}$$

## Solution of Circular Section using Prandtl's Method

$$\text{Eq: of circle, } [x^2 + y^2 = R^2] \text{ or } [x^2 + y^2 - R^2 = 0]$$

Prandtl's stress function for circular cross section

$$\phi = m(x^2 + y^2 - R^2) \quad | m = \text{constant}$$

Step 1 Poisson's eq

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\alpha$$

$$2m + 2m = -2G\alpha$$

$$4m = -2G\alpha$$

$$m = -\frac{G\alpha}{2}$$

$$\phi = -\frac{G\alpha}{2}(x^2 + y^2 - R^2)$$

$$\left| \begin{array}{l} \frac{\partial \phi}{\partial x} = m \times 2x \quad \frac{\partial^2 \phi}{\partial x^2} = 2m \\ \frac{\partial^2 \phi}{\partial y^2} = m \times 2y \quad \frac{\partial^2 \phi}{\partial y^2} = 2m \end{array} \right.$$

Step 2 Torque

$$T = 2 \iint \phi dxdy = 2 \iint -\frac{G\alpha}{2} (x^2 + y^2 - R^2) dxdy$$

$$T = -G\alpha \int_0^R (x^2 - R^2) 2\pi x dx$$

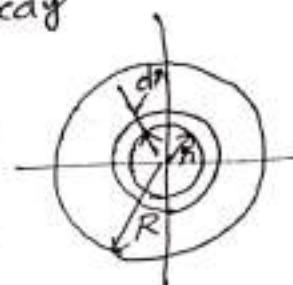
$$T = -G\alpha \times 2\pi \left[ \frac{x^4}{4} - R^2 \left( \frac{x^2}{2} \right) \right]_0^R$$

$$T = -2\pi G\alpha \left[ \frac{R^4}{4} - \frac{R^4}{2} \right]$$

$$T = -\frac{2\pi G\alpha R^4}{4}$$

$$\text{when } R = \frac{d}{2}$$

$$\rightarrow T = G\alpha \times 2\pi \frac{d^4}{16} = \frac{G\alpha \pi d^4}{32} \Rightarrow T = G\alpha J$$



Step 3 Angle of Twist

$$T = G\alpha J \Rightarrow \theta = \frac{T}{GJ}$$

Step 4 Resultant Shear Stress

$$\tau = \sqrt{\tau_{yz}^2 + \tau_{xz}^2}$$

$$\tau = \sqrt{(G\alpha z)^2 + (-G\alpha y)^2}$$

$$\tau = G\alpha \sqrt{x^2 + y^2} \Rightarrow \boxed{\tau = G\alpha R}$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = G\alpha x$$

$$\tau_{xz} = -\frac{\partial \phi}{\partial y} = -G\alpha y$$

Step 5 Torsional Equation

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\boxed{\tau_{max} = \frac{TR}{J}}$$

$| GJ = \text{Torsional Rigidity}$

$\theta = \text{Angle of twist/unit length}$

# Torsion of Elliptical Section/Bar using Prandtl's Method

## Torsion of Elliptical Bar

university Question ① For a bar of elliptical cross section with major and minor dimension  $2a$  and  $2b$ . Determine

- Stress function
- Component of stress
- Resultant stress  $\tau_{\max}$  value

a)  $\frac{\partial \phi}{\partial z}$  integrated

b) Torque equation

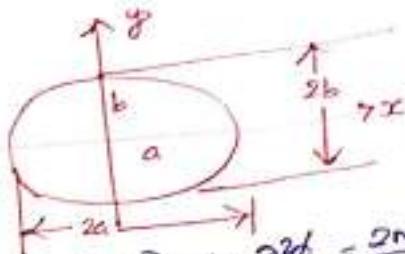
## Solution

Ellipse having major and minor axes  $2a$  &  $2b$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Prandtl's Stress Function

$$\phi = m \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad \text{①}$$



### Step ① Stress Function

$$\text{Poisson's eq. } \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\Theta$$

$$\text{Sub: eq. } \frac{2m}{a^2} + \frac{2m}{b^2} = -2G\Theta$$

$$2m \left[ \frac{1}{a^2} + \frac{1}{b^2} \right] = -2G\Theta \Rightarrow m \left( \frac{a^2 + b^2}{a^2 b^2} \right) = -G\Theta \Rightarrow$$

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= m \left( \frac{2x}{a^2} \right) \Rightarrow \frac{\partial^2 \phi}{\partial z^2} = \frac{2m}{a^2} \\ \frac{\partial \phi}{\partial y} &= m \left( \frac{2y}{b^2} \right) \Rightarrow \frac{\partial^2 \phi}{\partial y^2} = \frac{2m}{b^2} \end{aligned}$$

$$m = -\left[ \frac{a^2 b^2}{a^2 + b^2} \right] G\Theta \quad \text{GIC}$$

$$\text{Subeq. ①} \quad \phi = -\left[ \frac{a^2 b^2}{a^2 + b^2} \right] \left[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right] G\Theta$$

Prandtl's Stress Function

### Step ② Component of Stress

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -\left[ \frac{a^2 b^2}{a^2 + b^2} \right] \left[ \frac{2y}{b^2} \right] G\Theta = -\left( \frac{2a^2}{a^2 + b^2} \right) G\Theta y$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial z} = \frac{a^2 b^2}{a^2 + b^2} \left( \frac{2x}{a^2} \right) G\Theta = \left( \frac{2b^2}{a^2 + b^2} \right) G\Theta x$$

$$\text{Resultant Stress} \Rightarrow \tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \sqrt{\left( \frac{-2a^2}{a^2 + b^2} G\Theta y \right)^2 + \left( \frac{2b^2}{a^2 + b^2} G\Theta x \right)^2}$$

$$\tau_R = \frac{2G\Theta}{a^2 + b^2} \sqrt{a^2 y^2 + b^2 x^2}$$

$$\tau_R = \frac{2G\Theta}{a^2 + b^2} \sqrt{a^2 y^2 + b^2 \left( a^2 \left( \frac{b^2 - y^2}{b^2} \right) \right)}$$

$\tau_{\max}$  when  $y=b$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \Rightarrow \frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}$$

$$x^2 = a^2 \left( \frac{b^2 - y^2}{b^2} \right)$$

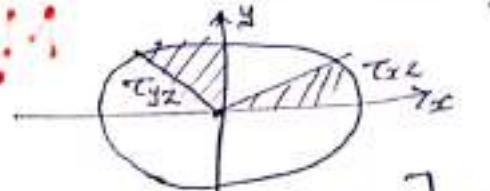
Ans

$$\tau_{\max} = \frac{2G\Theta}{a^2 + b^2} \sqrt{a^2 b^2} = \frac{2G\Theta a^2 b}{a^2 + b^2}$$

Step 3

J-integral

$$J = \frac{-1}{G10} \iint \left( \frac{x \partial \phi}{\partial x} + \frac{y \partial \phi}{\partial y} \right) dx dy$$



$$J = \frac{-1}{G10} \iint \left[ x \left( \frac{-a^2 b^2}{a^2 + b^2} \right) \frac{\partial \phi}{\partial x} G10 + y \left( \frac{-a^2 b^2}{a^2 + b^2} \right) \frac{\partial \phi}{\partial y} G10 \right] dx dy$$

$$J = \frac{-1}{G10} \left[ \iint \left( \frac{-b^2 \partial x^2 G10}{a^2 + b^2} \right) + \left( \frac{-a^2 \partial y^2 G10}{a^2 + b^2} \right) dx dy \right]$$

$$J = \frac{2G10}{G10(a^2 + b^2)} \left[ b^2 \iint x^2 dx dy + a^2 \iint y^2 dx dy \right]$$

$$J = \frac{2}{a^2 + b^2} \left[ b^2 \frac{\pi a b^3}{4} + a^2 \frac{\pi a b^3}{4} \right] \Rightarrow J = \frac{2}{a^2 + b^2} \times \frac{1}{4} [\pi b^3 a^3 + \pi a^3 b^3]$$

$$\boxed{J = \frac{\pi a^3 b^3}{a^2 + b^2}}$$

$$\begin{aligned} T &= G10 J \\ T &= G10 \times \frac{\pi a^3 b^3}{a^2 + b^2} \end{aligned}$$

$$I_x = \frac{\pi a b^3}{4}$$

$$I_y = \frac{\pi b a^3}{4}$$

Step 4 Angle of twist ( $\theta$ )

$$T = G10 J$$

$$\boxed{\theta = \frac{T}{G10} = \frac{T}{G10 \times \frac{\pi a^3 b^3}{a^2 + b^2}}}$$

displacement in z

$$w = \theta \phi$$

$$w = \frac{T(b^2 - a^2)}{\pi G10 a^3 b^3}$$

$$\phi = \frac{-T}{\pi a b} \left[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right]$$

Step 5 Warping Function ( $\psi$ )

$$\tau_{yz} = G10 \left( \frac{\partial \psi}{\partial y} + x \right) = \left( \frac{2b^2}{a^2 + b^2} \right) G10 x$$

$$x + \frac{\partial \psi}{\partial y} = \frac{2b^2 x}{a^2 + b^2} \Rightarrow \frac{\partial \psi}{\partial y} = \frac{2b^2 x}{a^2 + b^2} - x = \frac{2b^2 x - a^2 x - b^2 x}{a^2 + b^2} = \frac{b^2 x - a^2 x}{a^2 + b^2}$$

$$\boxed{\frac{\partial \psi}{\partial y} = \frac{(b^2 - a^2)x}{a^2 + b^2}}$$

Integrating

$$\boxed{\psi = \frac{(b^2 - a^2)}{a^2 + b^2} xy + C}$$

# MEMBRANE ANALOGY

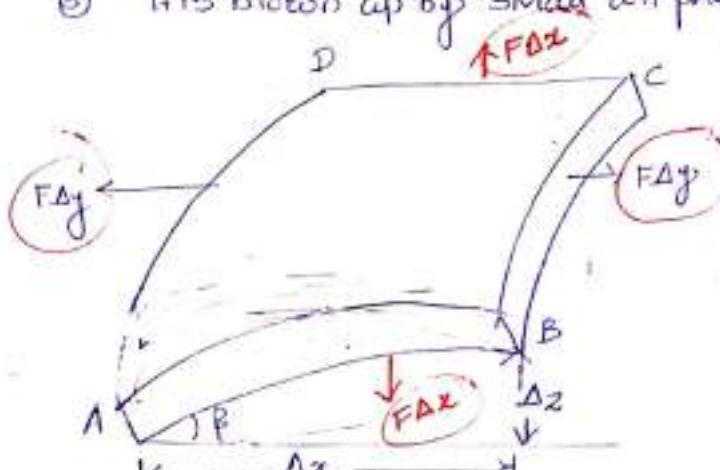
- \* An analogy helps in developing a difficult problem like torsion
- \* It was introduced by Prandtl & it is used to determine stresses in a section with negligible thickness

Eg. Scap film

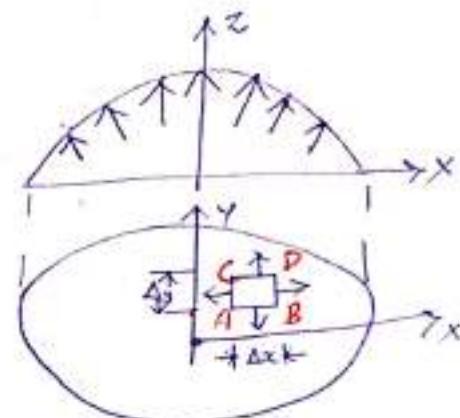
- \* A membrane is a 2D equivalent of strings, H carries load through constant tensile force

## Assumption

- ① The membrane is thin and weightless
- ② It is initially stretched with uniform forces per unit length
- ③ It is blown up by small air pressure



$$\text{Slope } \beta = \frac{\Delta z}{\Delta x}$$



## Major Features of Membrane Analogy

- ( $z=0$ ) ① Height of member  $z$  are numerically equal to stress function of  $(T=2V)$
- ( $T=2V$ ) ② The twisting moment is equal to twice the volume of membrane  $T_{zz} \cdot \frac{\partial z}{\partial y}$
- ③ Slope of membrane at a point is equal to shear stress at that point.  $T_{zx} = \frac{\partial z}{\partial y}, T_{zy} = -\frac{\partial z}{\partial x}$

## Derive Equilibrium eq of Membrane Analogy

### FACE AD

$$\left\{ \begin{array}{l} \text{Force} = F\Delta y \text{ (inclined } \beta \text{ with } x\text{-axis)} \\ \text{Slope} = \tan \beta = \frac{\partial z}{\partial x} \quad \text{Force} \times \text{slope} \\ \text{Component } z \text{ direction} = -F\Delta y \times \frac{\partial z}{\partial x} \end{array} \right.$$

### FACE BC

$$\left\{ \begin{array}{l} \text{Force} = F\Delta y \text{ (inclined } \beta + \Delta \beta \text{ to } x\text{-axis)} \\ \text{Slope of BC} = \beta + \frac{\partial \beta}{\partial z} \Delta z = \beta + \frac{\partial}{\partial z} \left( \frac{\partial z}{\partial x} \right) \Delta x \\ \text{Component in } z \text{ direction} = F\Delta y \left( \frac{\partial z}{\partial x} + \frac{\partial}{\partial z} \left( \frac{\partial z}{\partial x} \right) \Delta x \right) \end{array} \right.$$

### FACE AB

$$\left\{ \begin{array}{l} \text{Force} = -F\Delta x \\ \text{Component } z \text{ direction} = \left( -F\Delta x \times \frac{\partial z}{\partial y} \right) \end{array} \right.$$

### Face CD

$$\left\{ \begin{array}{l} \text{Force} = F\Delta x \\ z \text{ direction} = F\Delta x \left( \frac{\partial z}{\partial y} + \frac{\partial}{\partial z} \left( \frac{\partial z}{\partial y} \right) \Delta y \right) \end{array} \right.$$

Equilibrium resultant Forces in Z-direction

$$\begin{aligned} & -F\Delta y \frac{\partial z}{\partial x} + F\Delta y \left( \frac{\partial z}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \Delta x - F\Delta x \frac{\partial z}{\partial y} + F\Delta x \left( \frac{\partial z}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \Delta y \right) \right) \\ & -F\Delta y \frac{\partial z}{\partial x} + F\Delta y \frac{\partial z}{\partial x} + F\Delta y \frac{\partial x \frac{\partial^2 z}{\partial x^2}}{\partial x^2} - F\Delta x \frac{\partial z}{\partial y} + F\Delta x \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} \Delta_y \Delta_x + P\Delta x \Delta y = 0 \\ & F \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) \Delta x \Delta y = -P\Delta x \Delta y \end{aligned}$$

$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{P}{F}}$  Equilibrium eq of Membrane Analog

Analogy b/w poissisons eq

$$\text{poissisons eq } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G10 \quad \text{--- (2)}$$

$$\text{equating R.H.S of eq (1) & (2)} \Rightarrow -P/F = -2G10 \Rightarrow \boxed{\frac{P}{F} = 2G10}$$

$$\text{Multiply by } z \Rightarrow z \times \frac{P}{F} = 2G10 \times z$$

$$z = 2G10 \frac{F}{P} z$$

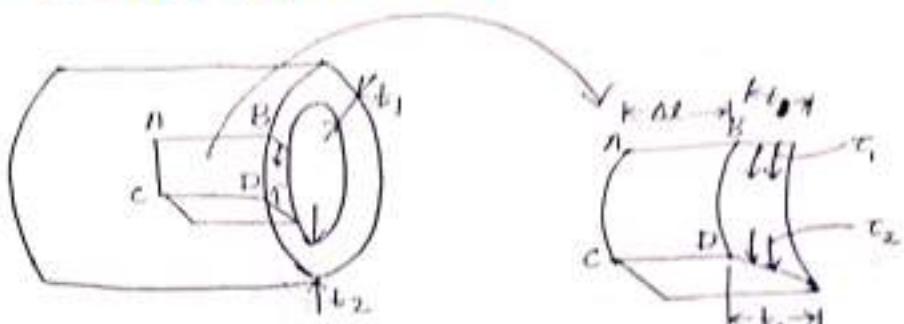
$$\text{Subeq(1)} \quad \frac{\partial^2}{\partial x^2} \left( 2G10 \frac{F}{P} z \right) + \frac{\partial^2}{\partial y^2} \left( 2G10 \frac{F}{P} z \right) = -2G10$$

$$\boxed{\phi = 2G10 \frac{F}{P} z} \quad \underline{\phi \propto z}$$

## Module - 6

### Torsion of thin walled tube (Fibre)

#### Shear Flow (q)



- A shaft is considered as thin-walled section when wall thickness less than  $\frac{1}{20}$  of its radius
- Consider thin-walled tube subjected to torsion, wall-thickness may vary from  $t_1$  to  $t_2$

$$[\text{Area of plate AB} = \Delta t b_1] \quad [\text{Area of plate CD} = \Delta t b_2]$$

$$\text{Shear force acting on AB} = T A = \tau_1 (\Delta t b_1)$$

$$\text{Shear force acting on CD} = T_2 (\Delta t b_2)$$

$$\text{At equilibrium above c/s, equate } \Rightarrow \tau_1 \Delta t b_1 - \tau_2 \Delta t b_2 = 0$$

$$\tau_1 b_1 = \tau_2 b_2$$

$$\tau_1 b_1 = \tau_2 b_2 = q$$

{ The product of shear stress and thickness at any section is always a constant called Shear flow (q) }

#### Determination of Torque AND shear

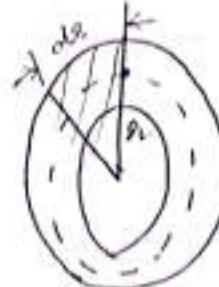
$$\begin{aligned} \text{Torque} &= \text{Force} \times \perp \text{ distance} \\ &= (T A) R = T (2\pi R^2) R = T \cancel{2\pi} R^3 \end{aligned}$$

$$T = R A T$$

or

$$T = 2 A R$$

} This is Bredt-Baudo formula



[Derive an eq for angle of twist per unit length for thin-walled tube subjected to torque T]

#### Determination of angle of twist

$$W = \frac{1}{2} P S \Rightarrow \theta = \frac{1}{2} (\tau A) (Y I)$$

$$\Rightarrow \theta = \frac{1}{2} (\tau Y V)$$

$$\Rightarrow \theta = \frac{1}{2} \left( \tau \times \frac{\tau}{G_I} \times V \right)$$

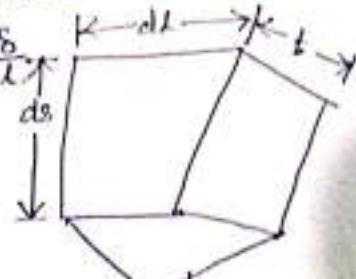
$$\Rightarrow \theta = \frac{1}{2} \left( \frac{T^2 V}{G_I} \right) \Rightarrow \theta = \frac{1}{2} \frac{\tau^2}{G_I} (d\sigma d\theta)$$

$$\Rightarrow \theta = \frac{1}{2 G_I} \left( \frac{\tau^2}{4 \pi R^2 t^2} \right) (d\sigma d\theta)$$

$$Y \text{ shear stress} = \frac{8}{3} \frac{R}{L}$$

$$A_l = \text{Volume} = V$$

$$G_I = \frac{I}{Y} \Rightarrow Y = \frac{I}{G_I}$$



$$\begin{aligned} T &= 2 A R \\ &= T \frac{2}{2 A E} \end{aligned}$$

$$U = \frac{T^2}{8G A^2 t} \cdot dA \cdot dt \quad \text{--- (1)}$$

$$\text{Also } U = \frac{1}{2} T \theta \quad \text{--- (2)}$$

$$\text{Equate eq (1) & (2)} \Rightarrow \frac{T^2}{8G A^2 t} dA \cdot dt = \frac{1}{2} T \theta$$

$$\Rightarrow \theta = \frac{T}{4GA^2} \left( \frac{dA}{t} \right) \quad \mid dA = 1$$

Angle of twist / unit length

$$\theta = \frac{T}{4GA^2} \int \frac{dA}{t} \quad \text{or} \quad \theta = \frac{q}{2AGI} \int \frac{ds}{t}$$

$$\begin{aligned} T &= \frac{T}{2At} \\ T &= 2Ag \end{aligned}$$

T = Torque  
A = Area

- Ques ① Design of a hollow Al section as shown in fig. If shear stress of material is 35000 kPa, G =  $157.5 \times 10^6$  N/mm²

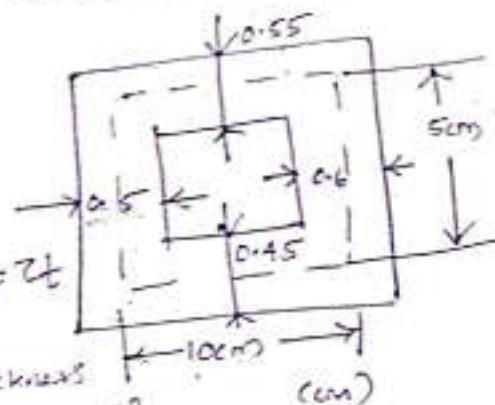
Ans

- (a) Find Torque transmitted  
(b) Find angle of twist / unit length

Solution

$$\begin{aligned} (a) \quad T &= 2qA = 2A(\tau_t) \\ &= 2 \left[ 10 \times 10^{-2} \times 5 \times 10^{-2} \times 157500 \right] \\ &= \underline{\underline{1575 \text{ Nm}}} \end{aligned}$$

$$\begin{aligned} q_t &= \text{Permissible shear flow} \\ &= \text{Shear stress} \times \text{unit thickness} \\ &= 35000 \times 0.45 \times 10^{-2} \\ &= \underline{\underline{157500 \text{ Nm}}} \\ A &= 10 \times 10^{-2} \times 5 \times 10^{-2} \text{ m}^2 \end{aligned}$$



- (b) Angle of twist / unit length

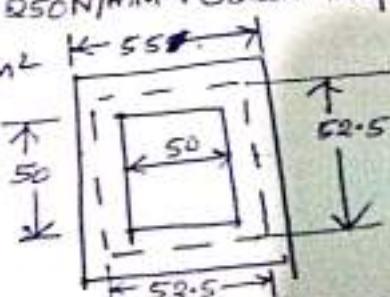
$$\begin{aligned} \theta &= \frac{q}{2AGI} \int \frac{ds}{t} \\ &= \frac{157500}{2 \times 157.5 \times 10^6 \times 50 \times 10^{-4}} \times 58.73 \\ &= \underline{\underline{0.0058 \text{ rad/m}}} \end{aligned}$$

$$\int \frac{ds}{t} = \frac{5}{0.5} + \frac{5}{0.6} + \frac{10}{0.55} + \frac{10}{0.45} = \underline{\underline{58.73}}$$

- Ques ② A shaft of square section as shown in figure is subjected to a twisting moment such that max. shear stress limited to 250 N/mm². Obtain torque and angle of twist if shaft 1.6m long. G = 70000 N/mm²

$$\begin{aligned} (a) \quad T &= 2qA = 2(\tau_t)A \\ &= 2(250 \times 2.5) \times 52.5^2 \\ &= 3415313 \text{ Nmm} \\ &= \underline{\underline{3415.313 \text{ Nm}}} \end{aligned}$$

$$\begin{aligned} \tau_t &= 250 \text{ N/mm}^2 \\ A &= 52.5 \times 52.5 \text{ mm}^2 \\ t &= 2.5 \text{ mm} \end{aligned}$$



- (b) Angle of twist / unit length

$$\begin{aligned} \theta &= \frac{q}{2AGI} \int \frac{ds}{t} = \frac{250 \times 2.5}{2 \times 52.5^2 \times 70000} \times 84 = 1.36 \times 10^{-4} \text{ rad/m} \\ \text{Angle of twist } \theta &= 1.36 \times 10^{-4} \times 1.6 = \underline{\underline{0.218 \text{ rad}}} \end{aligned}$$

$$u = \frac{T^2}{8GtA^2} ds dl \quad \text{--- (1)}$$

$$\text{Also } u = \frac{1}{2} T \theta \quad \text{--- (2)}$$

$$\text{Equate eq. (1) & (2)} \Rightarrow \frac{T^2}{8GtA^2} ds dl = \frac{1}{2} T \theta$$

$$\Rightarrow \theta = \frac{T}{4GtA^2} \left( \frac{ds}{l} \right) \quad \left| \frac{dl=1}{ds} \right.$$

Angle of twist / unit length

$$\theta = \frac{T}{4GtA^2} \int \frac{ds}{l} \quad \text{or} \quad \theta = \frac{q}{2AG} \int \frac{ds}{l}$$

$$T = \frac{T}{2At} \\ T = 2At$$

T = Torque  
A = Area

- Q.1 Design of a hollow Al Section as shown in fig. If shear stress of material is 35000 kPa, G = 157.5 x 10<sup>6</sup> kPa

Ans

- (a) Find Torque transmitted  
(b) Find angle of twist / unit length

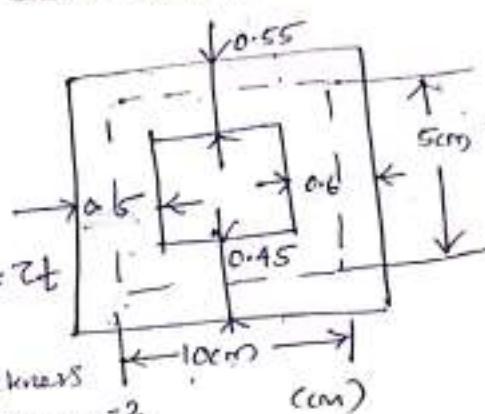
Solution

$$(a) T = 2qA = 2A(\tau t)$$

$$= 2 \left[ 10 \times 10^{-2} \times 5 \times 10^{-2} \times 157500 \right]$$

$$= 1575 \text{ Nm}$$

$$\begin{aligned} q &= \text{Permissible shear flow} \\ &= \text{Shear Stress} \times \text{Min thickness} \\ &= 35000 \times 10^3 \times 0.45 \times 10^{-2} \\ &= 157500 \text{ Nm} \\ A &= 10 \times 10^{-2} \times 5 \times 10^{-2} \text{ m}^2 \end{aligned}$$



- (b) Angle of twist / unit length

$$\theta = \frac{q}{2AG} \int \frac{ds}{l}$$

$$= \frac{157500}{2 \times 157.5 \times 10^6 \times 50 \times 10^{-4}} \times 58.73$$

$$= 0.0058 \text{ rad/m}$$

$$\int \frac{ds}{l} = \frac{5}{0.5} + \frac{5}{0.6} + \frac{10}{0.55} + \frac{10}{0.45} = 58.73$$

- Q.2 A shaft of square section as shown in figure is subjected to a twisting moment such that max. shear stress limited to 250N/mm<sup>2</sup>. Obtain torque and angle of twist. If shaft 1.6m long. G = 70000N/mm<sup>2</sup>

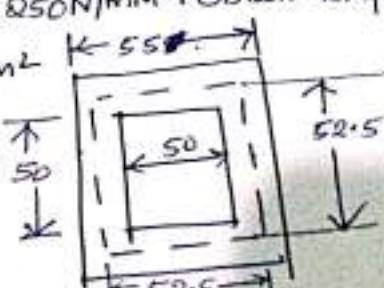
$$(a) T = 2qA = 2(\tau t)A$$

$$= 2(250 \times 2.5) \times 52.5^2$$

$$= 3415313 \text{ Nmm}$$

$$= 3415.313 \text{ Nm}$$

$$\begin{aligned} \tau &= 250 \text{ N/mm}^2 \\ A &= 52.5 \times 52.5 \text{ mm}^2 \\ t &= 2.5 \text{ mm} \end{aligned}$$



- (b) Angle of twist / unit length

$$\theta = \frac{q}{2AG} \int \frac{ds}{l} = \frac{250 \times 2.5}{2 \times 52.5^2 \times 70000} \times 84 = 1.36 \times 10^{-4} \text{ rad/mm}$$

$$\int \frac{ds}{l} = \frac{52.5}{2.5} + \frac{52.5}{2.5} + \frac{52.5}{2.5} + \frac{52.5}{2.5} \cdot 2t = 55 - 50 = 5$$

$$= 84$$

$$\text{Angle of twist } \theta = 1.36 \times 10^{-4} \times 1.6 = 0.218 \text{ rad} \quad \text{Ans}$$

③ A hollow section shown in figure designed for shear stresses of 40 MPa neglecting stress concentration. Find the twisting moment that can be taken up by the section and angle of twist. If the section is redesigned as hollow circular section of thickness 12 mm. Find diameter to take up the same twisting moment.

Solution

$$\tau_{max} = 40 \text{ MPa} = 40 \text{ N/mm}^2$$

$$t = 7 \text{ mm}$$

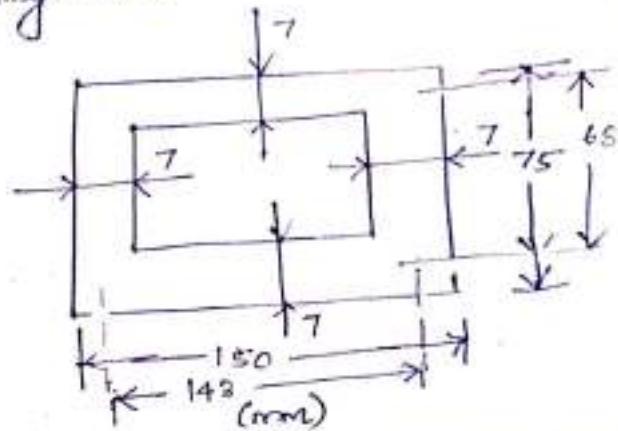
$$A = 143 \times 63 \text{ mm}^2$$

$$(a) T = 2qA = 2(\tau t)A = 2 \times (40 \times 7) (143 \times 63)$$

$$= \underline{\underline{5445440 \text{ Nmm}}}$$

$$\text{angle of twist/unit length } \theta = \frac{q}{2AG_1} \int \frac{ds}{t}$$

$$\theta = \frac{40 \times 7}{2(143 \times 63)G_1} \times 60.28 = \underline{\underline{0.8679 \text{ rad/s}}}$$



$$\int \frac{ds}{t} = \frac{143}{7} + \frac{143}{7} + \frac{68}{7} + \frac{68}{7} = \underline{\underline{60.28}}$$

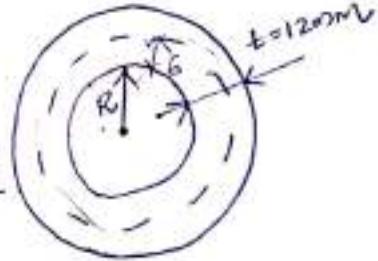
④ For circular section

$$T = 2qA$$

$$T = 2\tau t A$$

$$5445440 = 2 \times 40 \times 12 \times (\pi(R+6)^2) \rightarrow R = 36.4 \text{ mm}$$

$$\text{External dia} = 12 + 2 \times 36.4 + 12 = \underline{\underline{96.8 \text{ mm}}}$$



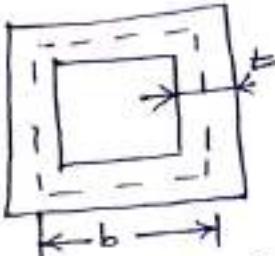
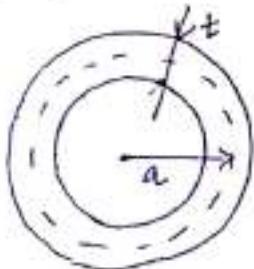
⑤ The figure shows the cross-section of two tubular rods. The thickness and circumference of two sections are equal. Find ratio of shear stresses

- (a) Equal twisting moment
- (b) Equal angle of twist

Solution

Given circumferential areas are equal

$$2\pi a = 4b \Rightarrow b = \frac{\pi a}{2} \quad t = 2b$$



$$\begin{aligned} \text{Area of square tube } A_s &= b^2 = \frac{\pi^2 a^2}{4} \\ \text{Area of circular tube } A_c &= \pi a^2 \end{aligned}$$

$$(a) T_c = T_s$$

$$\cancel{\frac{q_c}{q_s} A_c = \cancel{\frac{q_s}{q_s} A_s}} \Rightarrow \frac{q_c}{q_s} = \frac{A_s}{A_c} \rightarrow ①$$

$$\Rightarrow \frac{q_c}{q_s} = \frac{\pi^2 a^2 / 4}{\pi a^2} = \frac{\pi a^2 \times 1}{\pi a^2} = \underline{\underline{\frac{\pi}{4}}}$$

$$(b) \theta_c = \theta_s$$

$$\Rightarrow \frac{q_c}{2G_c A_c} \int \frac{ds}{t} = \frac{q_s}{2G_s A_s} \int \frac{ds}{t}$$

$$\Rightarrow \frac{q_c}{A_c} = \frac{q_s}{A_s}$$

$$\frac{q_c}{q_s} = \frac{A_c}{A_s} = \frac{1}{\pi}$$

# Torsion of Thin walled Multiple cell closed Section

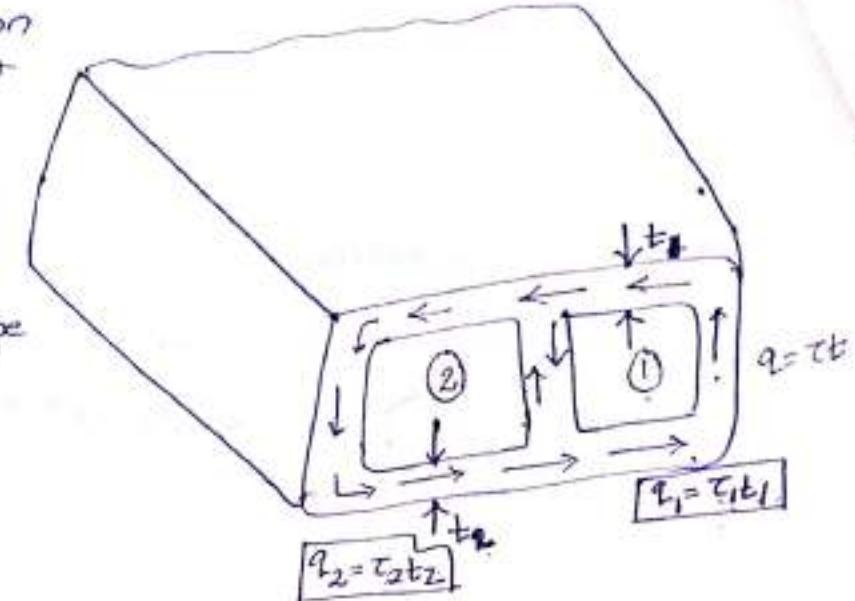
- \* Consider two cell section as shown in fig, for equilibrium of section at the junction  $T = 2Aq$

$$T = 2q_1 A_1 + 2q_2 A_2$$

- \* To find angle of twist, we have to consider cell 1 and cell 2 separately, we get

$$\theta_1 = \frac{1}{2A_1 G_1} [q_1 q_1 - q_{12} q_2]$$

$$\theta_2 = \frac{1}{2A_2 G_2} [q_2 q_2 - q_{12} q_1]$$



For finding shear flows  $q_1$  and  $q_2$  equate

$$\theta_1 = \theta_2$$

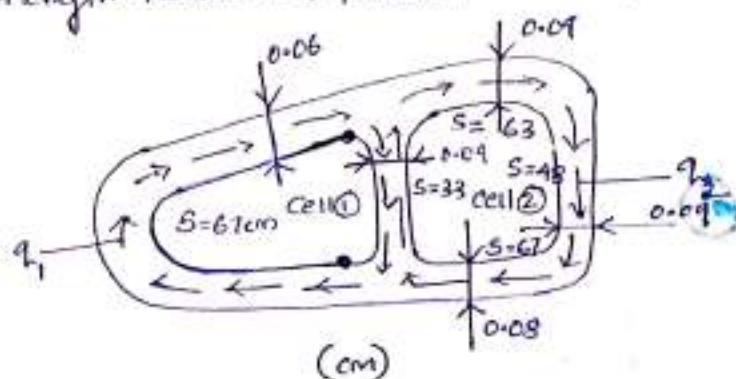
- The fig. Shows a two cell-tubular section having an interior web with an external torque of 10,000 Nmm. Determine the value of shear force and angle of twist per unit length. Assume  $A_1 = 680 \text{ cm}^2$  and  $A_2 = 2000 \text{ cm}^2$

Solution

$$\text{Cell } ① \rightarrow q_1 = \frac{67}{0.06} + \frac{33}{0.09} = 1483.33$$

$$\text{Cell } ② \rightarrow q_2 = \frac{67}{0.08} + \frac{48}{0.09} + \frac{63}{0.09} + \frac{33}{0.09} = 2716.6$$

$$\text{For web } \rightarrow q_{12} = \frac{33}{0.09} = 366.6$$



$$T = 2q_1 A_1 + 2q_2 A_2 \rightarrow 10000 \times 10^2 = 2q_1 (680) + 2q_2 (2000)$$

$$185 \times 10^2 = 136q_1 + 50q_2 \quad ①$$

$$\theta_1 = \theta_2 \Rightarrow \frac{1}{2G_1 A_1} [q_1 q_1 - q_{12} q_2] = \frac{1}{2G_2 A_2} [q_2 q_2 - q_{12} q_1]$$

$$\frac{1}{680} [1483.33 q_1 - 366.66 q_2] = \frac{1}{2000} [2716.6 q_2 - 366.66 q_1]$$

$$21.81 q_1 - 5.39 q_2 = 13.58 q_2 - 1.83 q_1$$

$$23.64 q_1 = 18.88 q_2 \Rightarrow q_2 = 1.34 q_1 \quad ②$$

$$\text{From } ① \text{ & } ② \quad 18500 = 136q_1 + 50 \times 1.34 q_1$$

$$q_1 = 149.8 \text{ N/cm}$$

$$q_2 = 1.34 q_1 = 199.4 \text{ N/cm}$$

### Angle of twist/unit length

$$\theta = \frac{1}{2Gt \times 10^6} [1483.3 \times 148.5 - 3666 \times 199] = \frac{109.64}{Gt} \text{ rad/cm}$$

- (2) The cacos section of a Steel Section as shown in fig with uniform thickness 1.25cm and stress due to twisting do not exceed 350 MPa.

(a) Max Torque allowable

(b) Twist per unit length

(c) Shear stress in middle web

Solution

$$T = 350 \times 10^6 \text{ N/m}^2 = 350 \times 10^6 \times 10^{-4} \text{ N/cm}^2 = 350 \times 10^2 \text{ N/cm}^2$$

$$\text{Cell } ① \rightarrow q_1 = \frac{11}{1.25} + \frac{12}{1.25} + \frac{12}{1.25} + \frac{11}{1.25} = 36.8 \text{ cm}^2$$

$$\text{Cell } ② \rightarrow q_2 = \frac{12.5}{1.25} + \frac{12.5}{1.25} + \frac{11}{1.25} + \frac{11}{1.25} = 37.6 \text{ cm}^2$$

$$\text{Web} \rightarrow q_{12} = \frac{11}{1.25} = 8.8$$

$$T = 2q_1 A_1 + 2q_2 A_2 \\ = 2q_1(12 \times 11) + 2q_2(12.5 \times 11) \Rightarrow T = 264q_1 + 275q_2 \quad ①$$

$$q_1 = q_2$$

$$\frac{1}{2GtA_1} [q_1 q_1 - q_{12} q_2] = \frac{1}{2GtA_2} [q_2 q_2 - q_{12} q_1]$$

$$\frac{1}{12 \times 11} [36.8 q_1 - 8.8 q_2] = \frac{1}{12.5 \times 11} (37.6 q_2 - 8.8 q_1)$$

$$0.278q_1 - 0.066q_2 = 0.273q_2 - 0.064q_1$$

$$q_2 = 1.007q_1 \quad ②$$

$$\text{Since } q_2 > q_1 \Rightarrow T_{\max} \times t = q_2 \Rightarrow 350 \times 10^2 \times 1.25 = q_2 \Rightarrow q_2 = 43750 \text{ N/cm}$$

$$\text{From eq-} ② \quad q_2 = 1.007q_1 \Rightarrow 43750 = 1.007q_1$$

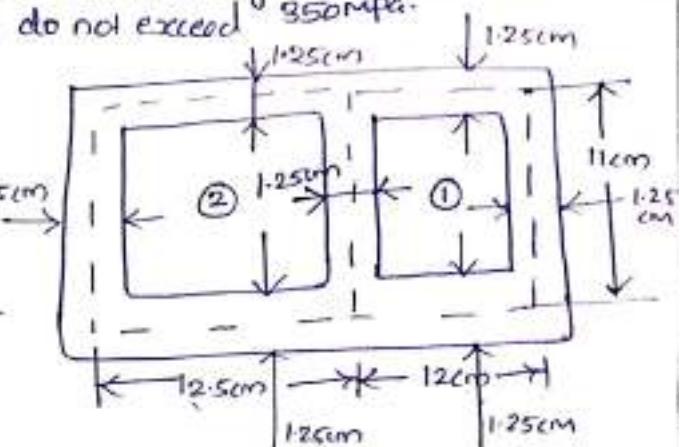
$$q_1 = \underline{\underline{43445.87 \text{ N/cm}}}$$

### Angle of twist

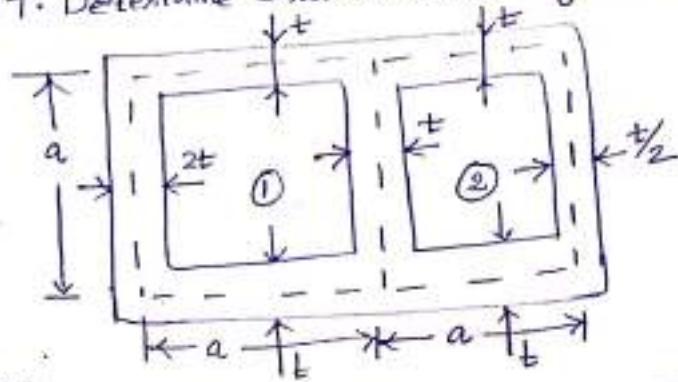
$$\theta = \frac{1}{2Gt(12 \times 11)} (36.8 \times 43445.81 - 8.8 \times 43750)$$

$$\theta = \underline{\underline{4597.7 \text{ rad/sec}}}$$

$$\tau = \frac{q}{t} = \frac{q_2 - q_1}{t} = \frac{43750 - 43445.87}{1.25} = \underline{\underline{243.3 \text{ N/cm}^2}}$$



③ The following shows two cell tubular section subjected to Torque T. Determine shear flow and angle of twist per unit length



Solution

$$q_1 = \frac{a}{t} + \frac{a}{2t} + \frac{a}{t} + \frac{a}{t} = \frac{7a}{2t}$$

$$q_2 = \frac{a}{t} + \frac{a}{t/2} + \frac{a}{t} + \frac{a}{t} = \frac{5a}{t}$$

$$q_{12} = \frac{a}{t}$$

$$T = 2q_1 A_1 + 2q_2 A_2 \Rightarrow T = 2q_1(axa) + 2q_2(axa) \Rightarrow [T = 2q_1 a^2 + 2q_2 a^2] \quad ①$$

$$\theta_1 = \theta_2$$

$$\frac{1}{2GIA_1} [a_1 q_1 - a_{12} q_2] = \frac{1}{2GIA_2} [a_2 q_2 - a_{12} q_1] \Rightarrow \frac{1}{axa} \left[ \frac{7a}{2t} q_1 - \frac{a}{t} q_2 \right] = \frac{1}{a^2} \left[ \frac{5a}{t} q_2 - \frac{a}{t} q_1 \right]$$

$$\frac{7a}{2t} q_1 - \frac{a}{t} q_2 = \frac{5a}{t} q_2 - \frac{a}{t} q_1 \Rightarrow \frac{7a}{2t} q_1 + \frac{a}{t} q_1 = \frac{5a}{t} q_2 + \frac{a}{t} q_2$$

$$\frac{9}{2} q_1 = 6 q_2 \Rightarrow q_2 = \frac{9}{12} q_1 = \frac{3}{4} q_1$$

$$\text{Sub: Value of } q_2 \text{ in eq } ① \quad T = 2q_1 a^2 + 2 \times \frac{3}{4} q_1 a^2 = 2a^2 (q_1 + \frac{3}{4} q_1) = \frac{7}{2} a^2 q_1$$

$$q_1 = \frac{2T}{7a^2} \quad \& \quad q_2 = \frac{3T}{14a^2}$$

Angle of twist/unit length  $\Rightarrow$

$$\theta_1 = \frac{1}{2GIA^2} \left[ \frac{7a}{2t} \times \frac{2T}{7a^2} - \frac{a}{t} \times \frac{3T}{14a^2} \right] = \frac{1}{2GIA^2} \left[ \frac{24T}{14ta} - \frac{3T}{14ta} \right]$$

$$\theta_1 = \frac{11T}{28GIA^2}$$

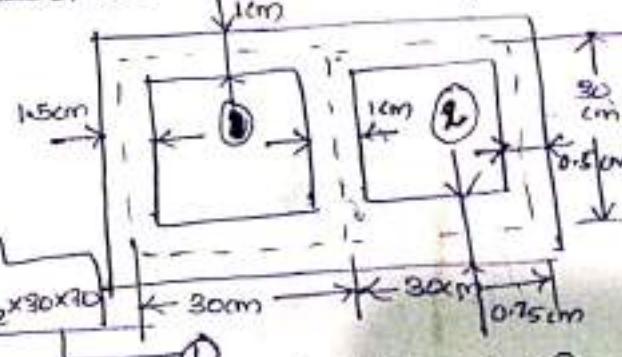
④ Find angle of twist/unit length when two cell tubular section as shown in fig 2 subjected to torque of 30,000 Nm. Value of  $G = 160 \times 10^9 \text{ N/m}^2$

$$\text{Cell ①} \Rightarrow a_1 = \frac{30}{0.75} + \frac{30}{1.5} + \frac{30}{1} + \frac{30}{1} = 120$$

$$\text{Cell ②} \Rightarrow a_2 = \frac{30}{0.75} + \frac{30}{1} + \frac{30}{0.5} + \frac{30}{1} = 160$$

$$\text{web} \Rightarrow a_{12} = \frac{90}{1} = 90$$

$$T = 2q_1 A_1 + 2q_2 A_2 \Rightarrow 30000 = 2q_1 \times 30 \times 20 + 2q_2 \times 30 \times 10$$



$$\theta_1 = \theta_2$$

$$\frac{1}{2GIA_1} [a_1 q_1 - a_{12} q_2] = \frac{1}{2GIA_2} [a_2 q_2 - a_{12} q_1] \Rightarrow \frac{1}{30 \times 30} [120q_1 - 30q_2] = \frac{1}{30 \times 30} [160q_1 - 30q_2]$$

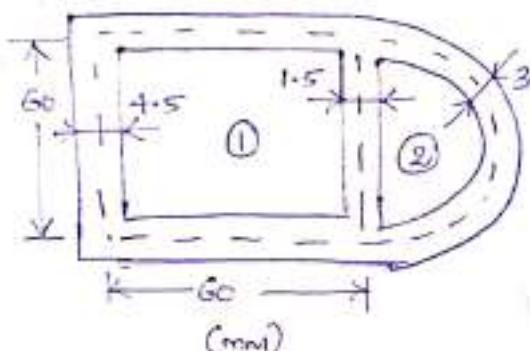
$$150q_1 = 190q_2 \Rightarrow q_1 = 1.267q_2 \quad \Rightarrow \quad ②$$

$$\text{Sub: eq ② in ①} \quad 30000 = 2 \times 1.267q_2 \times 30^2 + 2q_2 \times 30^2 \Rightarrow q_2 = 735.1860 \text{ N/cm}$$

$$\text{Angle of twist} \rightarrow \theta = \frac{1}{2GIA_1} [a_1 q_1 - a_{12} q_2] = \frac{1}{2 \times 160 \times 10^9 \times 30^2} [120 \times 9.91 \cdot 1.267 - 30 \times 735.1860] = 3.115351 \times 10^{-4}$$

5 A hollow thin walled torsion member has two compartment with cross sectional dimensions as given in figure. The material is an aluminium alloy having  $G_1 = 86 \text{ GPa}$ . Determine the torque  $T$  and angle of twist. If max shear stress is  $40 \text{ MPa}$

$$G_1 = 86 \text{ GPa} \\ = 86 \times 10^9 \text{ N/mm}^2 \\ = 86 \times 10^3 \text{ N/mm}^2$$



Solution

$$\text{For cell } ① \rightarrow a_1 = \frac{60}{4.5} + \frac{60}{4.5} + \frac{60}{1.5} + \frac{60}{4.5} = 60 \text{ mm}$$

$$\text{For cell } ② \rightarrow a_2 = \frac{60}{1.5} + \frac{2\pi \times 30}{2 \times 3} = 71.4 \text{ mm}$$

$$\text{For web} \rightarrow a_{12} = \frac{60}{1.5} = 40 \text{ mm}$$

$$A_1 = 60 \times 60 \\ = 3600 \text{ mm}^2 \\ A_2 = \frac{\pi r^2}{2} = \frac{\pi \times 30^2}{2} \\ = 1413 \text{ mm}^2$$

$$① T = 2A_1q_1 + 2A_2q_2 \Rightarrow T = 2 \times 3600q_1 + 2 \times 1413q_2 \quad ①$$

$$② G_1 = G_2 \Rightarrow \frac{1}{2A_1G_1} [a_1q_1 - a_{12}q_2] = \frac{1}{2A_2G_2} [a_2q_2 - a_{12}q_1]$$

$$\Rightarrow \frac{1}{2 \times 3600 \times 86} [80q_1 - 40q_2] = \frac{1}{2 \times 1413 \times 86} [71.4q_2 - 40q_1]$$

$$\Rightarrow 0.0224q_1 - 0.0112q_2 = 0.051q_2 - 0.0284q_1$$

$$0.0508q_1 = 0.0622q_2 \Rightarrow \frac{q_1}{q_2} = 1.224$$

Here  $\tau_{max} = 40 \text{ MPa}$  (given)

$$\text{Hence } (q_2 > q_1) \Rightarrow q_2 = \tau_{max} \times 3 = 40 \times 3 = 120 \text{ N/mm}$$

$$q_1 = 1.224 \times 120 = 146.4 \text{ N/mm}$$

Sub: value in eq-①

$$\Rightarrow \text{Torque } T = 2 \times 3600 \times 146.4 + 2 \times 1413 \times 120 = 1.3932 \text{ kNm}$$

$$\Rightarrow \text{Angle of twist} \Rightarrow \theta_1 = \frac{1}{2 \times 3600 \times 86 \times 10^3} (80 \times 146.4 - 40 \times 120) = 3.61 \times 10^{-5} \text{ rad/mm}$$

$$\Rightarrow \tau_1 = \frac{q_1}{t_1} = \frac{146.4}{4.5} = 32.5 \text{ N/mm}^2$$

$$\tau_{max} = \tau_2 = 40 \text{ N/mm}^2$$

$$\tau_3 = \frac{q_3}{t_3} = \frac{q_1 - q_2}{t_3} = \frac{26.4}{1.5} = 17.6 \text{ N/mm}^2$$

- Q. 6 The cross section of an Al shaft 7m long is shown in figure. Determine total angle of twist of the section and shear stresses in each part for applied torque of 400Nm. If  $G = 82 \text{ GPa}$

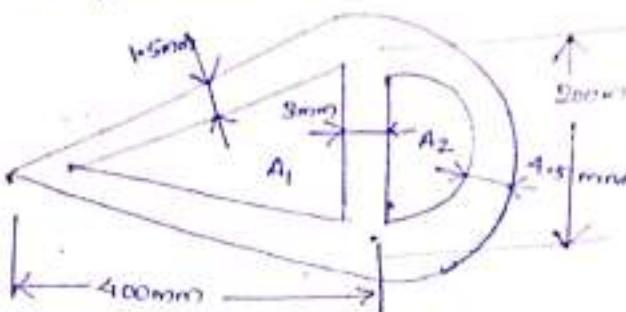
Q. 6

Solution

$$G = D \rightarrow T = 400 \text{ Nm}$$

$$G = 82 \text{ GPa}$$

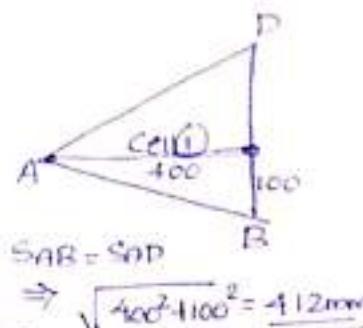
$$= 82 \times 10^9 \text{ N/mm}^2$$



$$q_1 = \text{For cell } ① \rightarrow \frac{412}{1.5} + \frac{412}{1.5} + \frac{200}{3} = \underline{\underline{616.007 \text{ mm}}}$$

$$q_2 = \text{For cell } ② \rightarrow \frac{200}{3} + \frac{\pi \times 100}{\pi \times 4.5} = \underline{\underline{136.45 \text{ mm}}}$$

$$q_{12} = \text{For web} \rightarrow \frac{200}{3} = \underline{\underline{66.667 \text{ mm}}}$$



$$① \quad T = 2q_1q_1 + 2q_2q_2 \Rightarrow \boxed{400 \times 10^3 = 2 \times 40000q_1 + 2 \times 15700q_2} \quad A_1 = \frac{1}{2}bh = \frac{1}{2}400 \times 200 = \underline{\underline{40000 \text{ mm}^2}}$$

$$② \quad \theta_1 = \theta_2 \Rightarrow \frac{1}{2A_1G} [q_1q_1 - q_1q_2] = \frac{1}{2A_2G} [q_2q_2 - q_1q_2] \quad A_2 = \frac{\pi d^2}{4} = \frac{\pi \times 100^2}{4} = \underline{\underline{15700 \text{ mm}^2}}$$

$$\Rightarrow \frac{1}{2 \times 40000} [616q_1 - 66.667q_2] = \frac{1}{2 \times 15700} [136.45q_2 - 66.667q_1] \quad \underline{\underline{15700 \text{ mm}^2}}$$

$$\Rightarrow 0.0154q_1 - 0.00167q_2 = 0.00869q_2 - 0.00425q_1$$

$$\Rightarrow 0.01965q_1 = 0.01036q_2 \Rightarrow \frac{q_1}{q_2} = \underline{\underline{0.527}}$$

$$\Rightarrow q_1 = 0.527q_2$$

$$\text{Sub: value of } q_1 \text{ in eq. } ① \quad 400 \times 10^3 = 80000 \times (0.527q_2) + 2 \times 15700 \times q_2$$

$$400 \times 10^3 = 73560q_2 \Rightarrow q_2 = \underline{\underline{5.438 \text{ N/mm}^2}}$$

$$q_1 = 0.527 \times 5.438 = \underline{\underline{2.866 \text{ N/mm}^2}}$$

$$T_1 = \frac{q_1}{t_1} = \frac{2.866}{1.5} = \underline{\underline{1.91 \text{ N/mm}^2}}, \quad T_2 = \frac{q_2}{t_2} = \frac{5.438}{4.5} = \underline{\underline{1.2 \text{ N/mm}^2}}$$

$$T_3 = \frac{q_3}{t_3} = \frac{q_1 - q_2}{t_3} = \frac{2.866 - 5.438}{3} = \underline{\underline{0.857 \text{ N/mm}^2}}$$

$$T_3 = \frac{q_3}{t_3} = \frac{q_2 - q_1}{t_3} = \frac{5.438 - 2.866}{3} = \underline{\underline{0.857 \text{ N/mm}^2}}$$

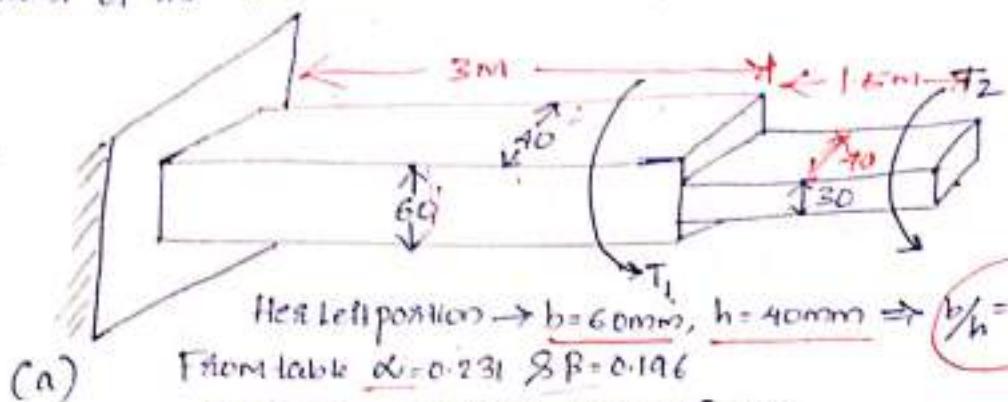
$$\theta_1 = \frac{1}{2A_1G} [q_1q_1 - q_1q_2] = \frac{1}{2 \times 40000 \times 82 \times 10^9} [616 \times 2.866 - 66.667 \times 5.438] = \underline{\underline{2.139 \times 10^{-7} \text{ rad/mm}^2}}$$

$$(l = 7 \text{ m}) \quad \text{Angular twist} = \frac{2.139 \times 10^{-7} \times 7000}{360^\circ} = \underline{\underline{1.4973 \times 10^{-5} \text{ rad}}} \text{ per meter}$$

Q7 A rod with rectangular cross-section is used to transmit torque to a machine frame (see figure). It has a width of 40mm. The first 3m length of rod has a depth of 60mm and the remaining 1.5m length has depth of 80mm. The rod is made of steel having  $G = 77.5 \text{ GPa}$ . Given  $T_1 = 750 \text{ Nm}$  &  $T_2 = 400 \text{ Nm}$ . Determine the maximum shear stress in the rod. Also determine the angle of twist of the free end.

$$\frac{b}{h} \propto \beta$$

1	0.205	0.141
2	0.210	0.166
3	0.231	0.196
4	0.231	0.214
5	0.246	0.224
6	0.258	0.219
7	0.267	0.263
8	0.282	0.281
9	0.299	0.299
10	0.307	0.307
11	0.313	0.313



$$T = T_1 + T_2 = 750 + 400 = 1150 \times 10^3 \text{ Nmm}$$

$$\tau_{\text{Max}} \text{ at left position} = \frac{T}{\alpha b h^2} = \frac{1150 \times 10^3}{0.231 \times 60 \times 40^2} = 51.86 \text{ N/mm}^2$$

$$\text{Right position } b = 40 \text{ } \& h = 80 \text{ } \& \frac{b}{h} = 0.5 \text{ } [K = 0.223, \beta = 0.178]$$

Hence Torque acting Left position

$$T = T_2 = 400 \times 10^3 \text{ Nmm}$$

$$\tau_{\text{Max}} \text{ at right position} = \frac{T}{\alpha b h^2} = \frac{400 \times 10^3}{0.223 \times 40 \times 80^2} = 44.83 \text{ N/mm}^2$$

$\tau_{\text{Max}}$  occurs at left position

(b) Angle of twist =  $\theta_{\text{left}} + \theta_{\text{right}} \Rightarrow$

$$= 0.059 + 0.040 = 0.099 \text{ rad. ans}$$

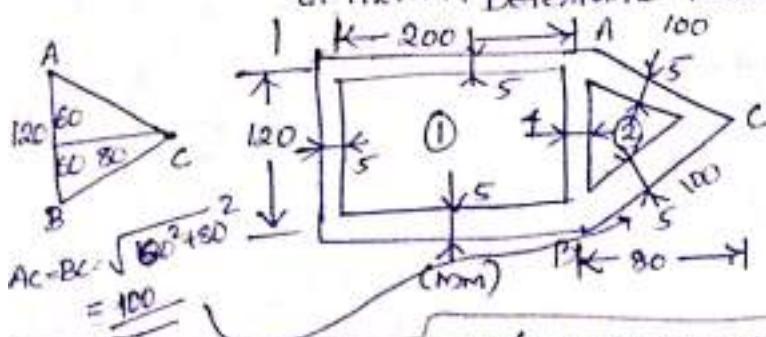
$$\theta_{\text{left}} = \frac{TL}{BGbh^3} = \frac{1150 \times 10^3 \times 3000}{0.196 \times 77.5 \times 10^3 \times 60 \times 40^3} = 0.059 \text{ radians}$$

$$\theta_{\text{right}} = \frac{TL}{BGbh^3} = \frac{400 \times 10^3 \times 1500}{0.178 \times 77.5 \times 10^3 \times 40 \times 80^3} = 0.040 \text{ radians}$$

university (8)

2019 Dec

The Al ( $G = 27 \text{ GPa}$ ) hollow thin walled torsion member has dimension as shown fig. If length is 3m. If the member subjected to torque of 11kNm. Determine Maximum Shear Stress & angle of twist



$$G = 27 \times 10^3 \text{ N/mm}^2$$

$$T = 11 \times 10^6 \text{ Nmm}, L = 3000 \text{ mm}$$

$$\text{cell ①} \rightarrow a_1 = \frac{120}{5} + \frac{200}{5} + \frac{120}{4} + \frac{200}{4} = 134 \text{ mm}$$

$$\text{cell ②} \rightarrow a_2 = \frac{120}{4} + \frac{100}{5} + \frac{100}{5} = 70 \text{ mm}$$

$$\text{web} \rightarrow a_{12} = \frac{120}{4} = 30 \text{ mm}$$

$$A_1 = 24000 \text{ mm}^2$$

$$A_2 = \frac{120 \times 80}{2} = 4800 \text{ mm}^2$$

$$\textcircled{1} \quad T = 2A_1a_1 + 2A_2a_2 \Rightarrow 11 \times 10^6 = 2 \times 24000a_1 + 2 \times 4800a_2$$

$$\textcircled{2} \quad \theta_1 = \theta_2 \Rightarrow \frac{1}{2A_1G} [a_1a_1 - a_{12}a_{12}] = \frac{1}{2A_2G} [a_2a_2 - a_{12}a_{12}]$$

$$\Rightarrow \frac{1}{24000} [134a_1 - 30a_2] = \frac{1}{4800} [70a_2 - 30a_1] \Rightarrow 0.01225a_1 = 0.14734 \Rightarrow 0.006a_1 - 0.00125a_2 = 0.14734$$

$$\textcircled{1} \quad a_1 = 12.024a_2$$

$$\text{sub: value eq-1} 11 \times 10^6 = 4800(12.024a_2) + 4800a_2$$

$$a_2 = 18747 \text{ N/mm}, a_1 = 225.33 \text{ N/mm}$$

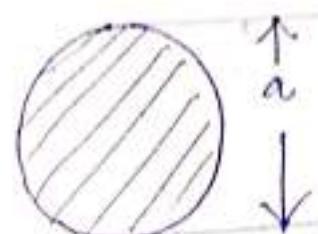
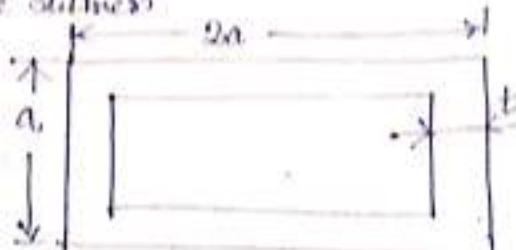
$$\tau_1 = a_1 / t_1 = \frac{225.33}{5} = \dots, \tau_2 = \frac{a_2}{t_2} = \frac{18747}{5} = \dots$$

$$T_3 = a_1 a_2 / 4$$

Ques (i) A thin walled box section  $2a \times a \times t$  is to be compared with a rectangular section having diameter ' $a$ ' shown in fig. Find the thickness ' $t$ ' so that both sections have:

(i) Same shear stress for same torque.

(ii) Same stiffness.



(i) Same shear stress for same torque.

$$T = 2qA - \text{for box section}$$

$$\text{where } A = 2a^2, q = \tau t \quad \therefore T = 2\tau t 2a^2 \quad \text{--- (1)}$$

$$\frac{T}{2} = \frac{\tau}{n} \Rightarrow T = \frac{\tau I}{n} - \text{for solid circular section}$$

$$\text{where } n = \frac{a}{2}, I = \frac{\pi}{32} d^4 = \frac{\pi}{32} a^4 \quad \therefore T = \frac{\tau \times \frac{\pi}{32} a^4}{\frac{a}{2}} \Rightarrow T = \frac{\tau \pi a^3}{16} \quad \text{--- (2)}$$

$$\text{Equate (1) \& (2)} \quad \therefore \frac{2\tau t 2a^2}{16} = \frac{\tau \pi a^3}{16} \Rightarrow t = \frac{\pi a}{64}$$

(ii) Same stiffness

$$\text{Stiffness} = \frac{T}{\theta}$$

$$\text{We know that } \theta = \frac{T}{4A^2G_1} \int \frac{ds}{t} \text{ for box} \Rightarrow \frac{T}{\theta} = \frac{4A^2G_1}{\int \frac{ds}{t}} \quad \text{--- (1)}$$

$$\text{Assume (1+1)} \quad \frac{T}{2} = \frac{G_1 \theta}{L} \Rightarrow \frac{T}{\theta} = G_1 \frac{L}{2}$$

$$\Rightarrow \frac{T}{\theta} = G_1 \times \frac{\pi}{32} a^4 \quad \text{--- (2)}$$

Equate eq (1) & (2)

$$\frac{4A^2G_1}{\int \frac{ds}{t}} = \frac{G_1 \pi a^4}{32}$$

$$\frac{4 \times 4a^4 \times G_1}{6a/t} = \frac{G_1 \pi a^4}{32}$$

$$A = 2a^2$$

$$\int \frac{ds}{t} = \frac{a}{t} + 2a + \frac{a}{t} + \frac{2a}{t}$$

$$\int \frac{ds}{t} = \frac{6a}{t}$$

$$\frac{16a^4 G_1 L}{6a/t} = \frac{G_1 \pi a^4}{32}$$

$$\Rightarrow t = \frac{6 \pi a}{16 \times 32}$$

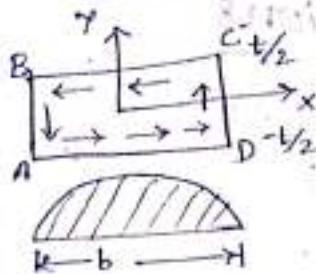
$$\text{or } t = \frac{3 \pi a}{4 \times 64}$$

Previous problem  
No: (8)

$$\theta_1 = \frac{1}{2A_1 G_1} [a_1 a_1 - a_1^2 a_2] = \frac{1}{2 \times 24000 \times 27 \times 10^3} \left[ \frac{13.4 \times 10^{-4} - 30 \times 18.747}{225.33} \right]$$

$$\text{Angle twist/unit length} = \theta_1 \times 3000 = \frac{\text{Radian/mm}}{\text{Radian}}$$

# Torsion of Thin Rectangular Section



Questions: Prove 1100 per sec on  
simplifies for thin rectangular  
section OR  
PROVE  $T = \frac{1}{3} G I_{GJ}$

- \* Here width  $b$  of rectangle much greater than (10 times) of its thickness  
It is called thin rectangle

- Consider thin rectangular section subjected to torque  $T$

Step 1 Assume  $\phi = \text{constant}$  for this case

$$\text{Poisson's eq} \rightarrow \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\alpha \quad [\text{Here } \phi \text{ varies across } y \text{ & is independent of } z]$$

$$\frac{\partial^2 \phi}{\partial y^2} = -2G\alpha \Rightarrow \frac{\partial \phi}{\partial y} = -2G\alpha y + c_1 \Rightarrow \phi = -2G\alpha \frac{y^2}{2} + c_1 y + c_2$$

$$\boxed{\phi = -G\alpha y^2 + c_1 y + c_2} \quad \text{--- (1)}$$

Apply Boundary condition, we get  $c_1 = 0$

$$\text{when } y = \pm \frac{t}{2}, \phi = 0 \Rightarrow 0 = -G\alpha \frac{t^2}{4} + c_1 t + c_2 \Rightarrow \boxed{c_2 = G\alpha \frac{t^2}{4}}$$

Sub: value of  $c_1$  &  $c_2$  in eq. (1)

$$\phi = -G\alpha y^2 + G\alpha \frac{t^2}{4} \Rightarrow \boxed{\phi = G\alpha \left( \frac{t^2}{4} - y^2 \right)}$$

Step 2 Shear stress

$$\tau_{yz} = \frac{\partial \phi}{\partial x} = 0, \tau_{zx} = \frac{\partial \phi}{\partial y} = G\alpha (-2y)$$

$\tau_{\max}$  when  $y = \pm \frac{t}{2}$

$$\boxed{\tau_{\max} = \pm G\alpha t}$$

$$\tau_{\text{res}} = \sqrt{\tau_{yz}^2 + \tau_{zx}^2} \\ = 2G\alpha y$$

Step 3

→ Torque

$$T = 2 \int \int \phi dz dy = 2 \int \int G\alpha \left( \frac{t^2}{4} - y^2 \right) dz dy = 2G\alpha \left[ \int_{-b/2}^{b/2} dz \right] \left[ \int_{-t/2}^{t/2} \left( \frac{t^2}{4} - y^2 \right) dy \right]$$

$$\Rightarrow 2G\alpha \left[ x \right]_{-b/2}^{b/2} \left[ \frac{t^2}{4} y - \frac{y^3}{3} \right]_{-t/2}^{t/2} \Rightarrow 2G\alpha \left[ \frac{b}{2} + \frac{b}{2} \right] \left[ \frac{t^2}{4} \times \frac{t}{2} - \frac{t^3}{24} \right]$$

$$\Rightarrow 2G\alpha [b] \left[ \frac{t^3}{8} - \frac{t^3}{24} \right] - \left[ -\frac{t^3}{8} + \frac{t^3}{24} \right] - \left[ \frac{t^2}{4} \times -\frac{t}{2} - \left( \frac{t^3}{24} \right) \right]$$

$$\Rightarrow 2G\alpha b \left[ \frac{2t^3}{8} - \frac{2t^3}{24} \right] \Rightarrow T = 2G\alpha b \left[ \frac{t^3}{4} - \frac{t^3}{12} \right] \Rightarrow 2G\alpha b \left[ \frac{2t^3}{12} \right]$$

$$\boxed{T = \frac{1}{3} G\alpha b t^3}$$

Step 4

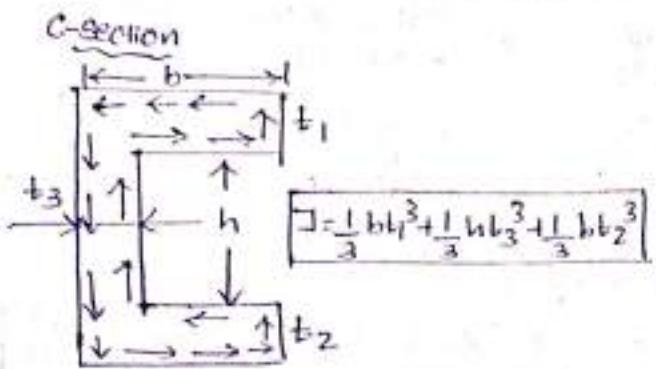
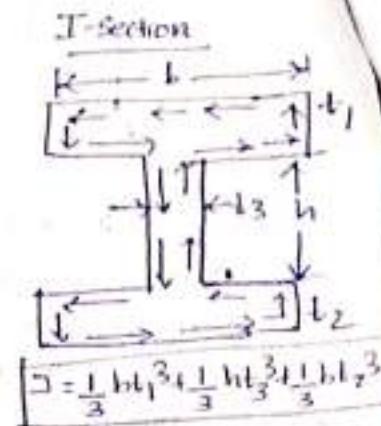
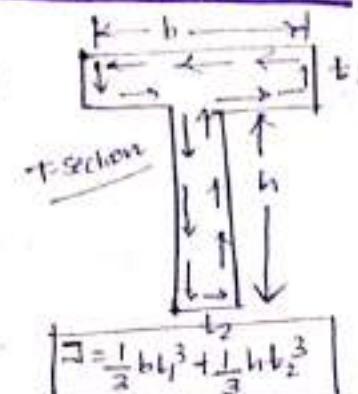
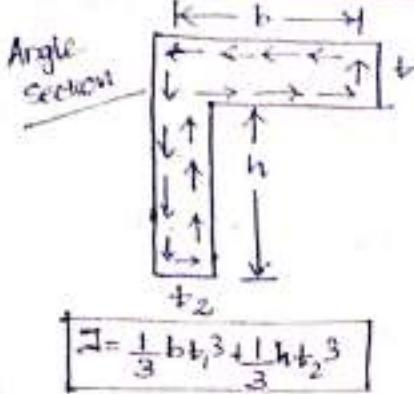
$$\theta = \frac{3T}{G I_{GJ}} \quad \& \quad \tau_{\max} = \pm G\alpha t = \frac{G \times 3T}{G I_{GJ}} \times \frac{t}{b}$$

$$\boxed{\tau_{\max} = \pm \frac{3T}{b t^2}}$$

Step 5

$$\text{I value } T = G I_{GJ} \Rightarrow I = \frac{T}{G I_{GJ}} = \frac{1}{3} \frac{G \alpha b t^3}{G I_{GJ}} = \frac{1}{3} b t^3$$

## Torsion of rolled Sections



① The open cross section like T-Section, I-Section, C-Section etc. are commoned under thin rectangular Section subjected to rolling.

② Here no. of thin rectangular arranged in the form of pie quised shape.

For thin rectangle  $T = \frac{1}{3}bt^3G_{10}$ .

General rectangles are two or more

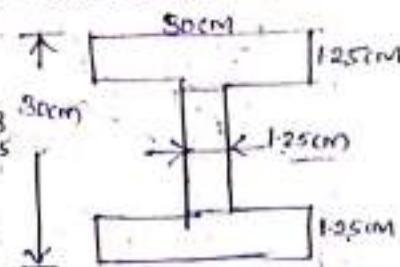
$$T = \frac{G_{10}}{3} [bt_1^3 + bt_2^3 + \dots]$$

For rectangle  $T = \sum_{i=1}^n \frac{G_{10}}{3} b_i t_i^3$  where  $G = \frac{3T}{G_{10}(b_i t_i^3)}$

- ① A 30cm I Section with flanges 3 each of 1.25cm thickness is subjected to torque of 4900 Nm. Find Max shear stress and angle of twist/unit length.

$$\theta = \frac{3T}{G(b_i t_i^3)} = \frac{3 \times 4900 \times 10^2}{G(30 \times 1.25^3 + 27 \times 1.25^3)}$$

$$\theta = \frac{8601.6}{G} \text{ rad/cm}$$



$$\tau_{max} = \frac{3T}{\frac{1}{2}b_i t_i^2} = \frac{3 \times 4900 \times 10^2}{30 \times 1.25^3 + 27 \times 1.25^2 + 30 \times 1.25^2}$$

$$= 10752 \text{ N/cm}^2 \checkmark$$

- (b) In order to reduce stress & angle of twist 1.25cm thick flat plates are welded on the side of section as dotted line. Find Torque & angle of twist.

Cell (1) & (2)  $A_1 = A_2 \Rightarrow$   $\frac{28.72 + 29.72}{1.25} + \frac{29.72 + 29.72}{1.25} = 64$

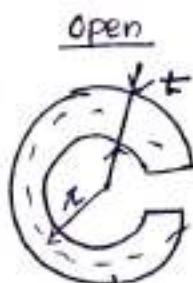
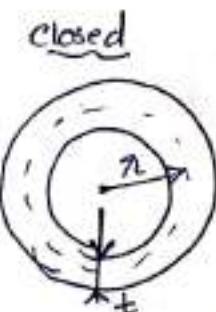
$$T = 2A_1 q_1 + 2A_2 q_2 \Rightarrow \text{Hence } (A_1 = A_2) (q_1 = q_2)$$

$$T = 4A_1 q_1 \Rightarrow T = 4T_i t_i \times A_1 \Rightarrow 4900 \times 10^2 = 4 \times T_i \times 1.25 \times \left( \frac{28.72 + 29.72}{2} \right)$$

University  
Question  
July 2017

\* Why closed section are having better torsional rigidity than open section. Briefly explain  
OR

Comparison b/w open & closed section



Case I

Torque  $\Rightarrow$

$$T_c = 2Aq$$

$$T_c = 2At$$

$$\boxed{T_c = 2\pi r^2 t} \text{ or } \boxed{T_c = 2\pi r^3 Gt} \quad (1)$$

$$\frac{I}{r} = Gt$$

$$T = GtR^2$$

$$T_o = \frac{Gtbt^3}{3}$$

$$\boxed{\frac{T_o = Gt \times 2\pi r^2 t^3}{3}} \quad (2)$$

$$b = 2\pi r$$

$$\frac{T_c}{T_o} = \frac{T_c}{T_o} = \frac{2\pi r^3 Gt}{3\pi r^2 t}$$

$$T_o = \frac{3T}{2\pi r^2 t} = \frac{3T}{2\pi r^2 t}$$

$$\text{Ratio of } \frac{T_c}{T_o} = \frac{2\pi r^3 Gt}{3\pi r^2 t} = 3\left(\frac{r}{t}\right)^2 \Rightarrow \text{Here } T_c = 3 \text{ times } T_o$$

$\therefore T_c \propto T_o \Rightarrow$  Torque  $\propto [G]$  (torsional rigidity)

$\therefore$  Closed Section have better torsional rigidity than open

$$\therefore \frac{\tau_o}{\tau_c} = \frac{3T}{2\pi r^2 t} = 3\left(\frac{r}{t}\right) \quad \left[ \begin{array}{l} \text{Here } T_o = 3 \text{ times } \left(\frac{r}{t}\right) \times T_c \\ T_o > T_c, \text{ Stress in open section} \\ \text{more than closed} \end{array} \right]$$

\* Angle of twist/unit length  $\Rightarrow$

$$f_{cls} = \frac{2\pi r}{t}$$

$$\theta_c = \frac{T}{4GIA^2} f_{cls}$$

$$\theta_o = \frac{3T}{Gbt^3} = \frac{3T}{G(2\pi r^2 t)}$$

$$\theta_c = \frac{T}{4GIA^2} \frac{2\pi r}{t}$$

$$\theta_c = \frac{T}{2GIA^3 t}$$

$$\frac{\theta_o}{\theta_c} = \frac{3T}{2GIA^3 t} \times \frac{4GIA^2}{3t} = \frac{3T}{2GIA^3 t} = \frac{t^2}{3r^2}$$

$$\text{Here } \theta_o = 3 \text{ times } \left(\frac{r^2}{t^2}\right) \theta_c$$

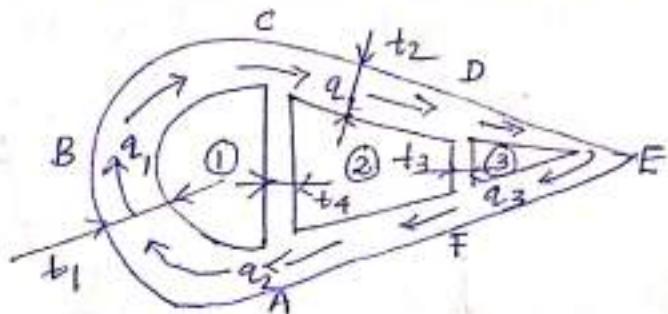
Angle of twist of open section more than closed

## Multiply Connected Section

A cross section is said to be multiply connected section, if it contains more than one closed curve in it.

Eg: A bar with one or more holes.

- \* When a hollow member having multiple cell in cross section is subjected to torsional loads, the torque applied is shared all the cell of the section.
- \* The governing eq are also similar to those who used two cells -

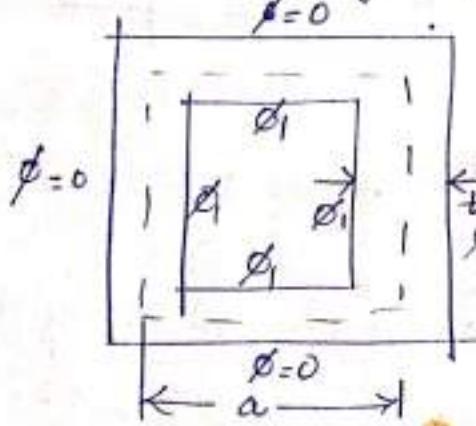


$$T = 2A_1\phi_1 + 2A_2\phi_2 + \dots \quad \text{--- (1)}$$

Also cell governing eq  $\oint \frac{\partial \phi}{\partial n} ds = 2G\phi A_i$

① Find the diagonal & also find shear stress along the cell

$$\phi = 0$$



$$\text{we have } \oint \frac{\partial \phi}{\partial n} ds = 2G\phi A_i$$

$$\Rightarrow \frac{\phi_1 - 0}{t} \times a + \frac{\phi_1 - 0}{t} \times a + \frac{\phi_1 - 0}{t} \times a + \frac{\phi_1 - 0}{t} \times a \\ \Rightarrow 4\phi_1 \times a = 2G\phi_1 a^2 = 2G\phi_1 a^2$$

$$\phi_1 = \frac{G\phi_1 a^2}{2}$$

$$T = 2A_1\phi_1 = 2A_1\phi_1 = \frac{\phi_1 a^2}{2} \left( \frac{G\phi_1 a^2}{2} \right) \Rightarrow T = a^3 G \phi_1$$

$$\tau = \frac{\partial \phi}{\partial r} = \frac{\phi_1}{t} = \frac{G\phi_1 a^2}{2t} = \frac{G\phi_1 a}{2}$$

$$J = a^3 t //$$

- ① Obtain the value of prandtl stress function at the inner surface  
 of bar (hollow circular bar)  
 ② Find the torque transmitted  
 ③ J-integral and shear stresses along the wall

Solution.

- (a) Boundary condition of  $\phi$  as

$$\int_i \frac{\partial \phi}{\partial n} ds = 2G\theta A_i$$

where  $\Rightarrow \frac{\partial \phi}{\partial n} \times 2\pi R = 2G\theta \pi R^2$

$$\boxed{\frac{\partial \phi}{\partial n} = G\theta R}$$

$$\left. \begin{aligned} \int_i ds &= 2\pi R \text{ (circumference)} \\ A_i &= \pi R^2 \end{aligned} \right\}$$

$$\frac{\partial \phi}{\partial n} = \frac{\phi_1 - 0}{t} = G\theta R$$

$\boxed{\phi_1 = G\theta R t}$   $\Rightarrow$  prandtl stress function

(b) Torque  $T = 2A_i \phi_1 + 2 \iint \phi dxdy$

(# thickness of bar small : double integral small  
compared to 1st term  
H Neglected)

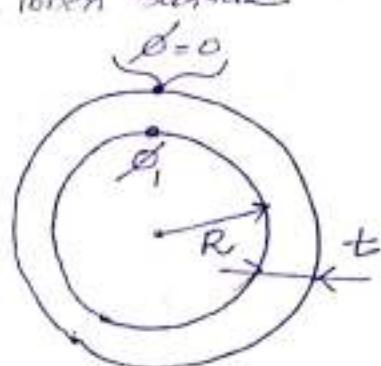
$$\begin{aligned} T &= 2A_i \phi_1 \\ &= 2\pi R^2 G\theta R t \\ &= 2\pi R^3 t G\theta \end{aligned}$$

(c)  $T = G\theta J$

$$T = G\theta \times 2\pi R^3 t$$

$$\underline{\underline{J \text{ integral}}} = \underline{\underline{2\pi R^3 t}}$$

Shear Stress  $\Rightarrow T = \frac{\partial \phi}{\partial n} = G\theta R$

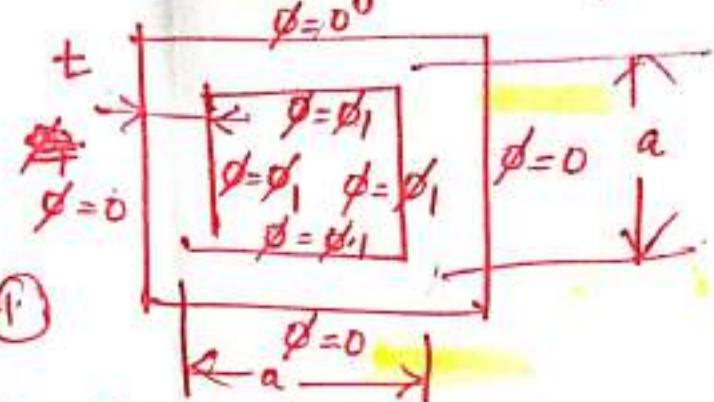


② Find D-integral of the section shown in fig. Assume that function varies linearly across C/s

$$\oint \frac{\partial \phi}{\partial n} ds = 2G_1 \alpha A_1$$

$$\frac{\partial \phi}{\partial n} \times 4a = 2G_1 \alpha a^2 A_1$$

$$\frac{\partial \phi}{\partial n} = \frac{G_1 \alpha a}{2}$$



$$\frac{\partial \phi}{\partial n} = \frac{\phi_1 - 0}{t} + \frac{\phi_1 - 0}{t} + \frac{\phi_1 - 0}{t} + \frac{\phi_1 - 0}{t} = \frac{4\phi_1}{t} \quad (2)$$

$$\frac{4\phi_1}{t} = \frac{G_1 \alpha a}{2} \Rightarrow \phi_1 = \frac{G_1 \alpha a t}{2}$$

$$T = 2A_1\phi_1 + 2 \iint \phi dxdy$$

$$T = 2a^2 \frac{G_1 \alpha a t}{2}$$

$$T = (G_1 \alpha a^3 t) \quad (T = G_1 \alpha J)$$

$$J = a^3 t$$

$$T = 2A_1\phi_1 + \iint \frac{\phi_1}{2} dxdy$$

$$T = 2A_1\phi_1 + \phi_1 \cdot 4(a+t) \quad \text{Diagram of a rectangle with width } a+t \text{ and height } a.$$

$$T = 2a^2 \times \frac{G_1 \alpha a t + G_1 \alpha a t + (a+t)^2}{2}$$

$$T = G_1 \alpha a t (a^2 + 2at + 2t^2)$$

$$T = G_1 \alpha J$$

(1)

= weight moment

I-integral  $T=6100 \text{ J}$

$$\therefore I = \alpha t(a^2 + 2at + 2t^2)$$

- ⑤ Find torsional shear stress and I-integral of the section shown in figure. Assume  $t \ll a$

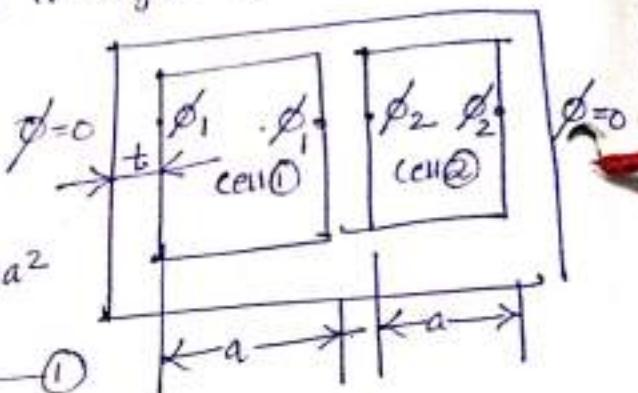
Solution

$$\oint \frac{\partial \phi}{\partial n} ds = 2G10A_1$$

Cell 0

$$\frac{\phi_1 - 0}{t} \times 3a + \frac{\phi_1 - \phi_2}{t} \times a = 2G10a^2$$

$$4\phi_1 - \phi_2 = 2G10at \quad \textcircled{1}$$



Cell 2

$$\frac{\phi_2 - 0}{t} \times 3a + \frac{\phi_2 - \phi_1}{t} \times a = 2G10a^2$$

$$4\phi_2 - \phi_1 = 2G10at \quad \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow 5\phi_1 - 5\phi_2 = 0$$

$$\phi_1 = \phi_2 \Rightarrow \text{Subeq } \textcircled{1}$$

$$3\phi_1 = 2G10at$$

$$\frac{1}{t} \sim \dots$$

Toque

$$T = 2A_1\phi_1 + 2A_2\phi_2 + 2 \int \frac{\partial \phi}{\partial r} dr$$

$t \ll a$

neglected

$$\begin{aligned} T &= 2A_1\phi_1 + 2A_2\phi_2 = 4A_1\phi_1 \\ &= 4a^2 \times \frac{2}{3} G10at = \frac{8G10a^3t}{3} \\ T &= G10J \end{aligned}$$

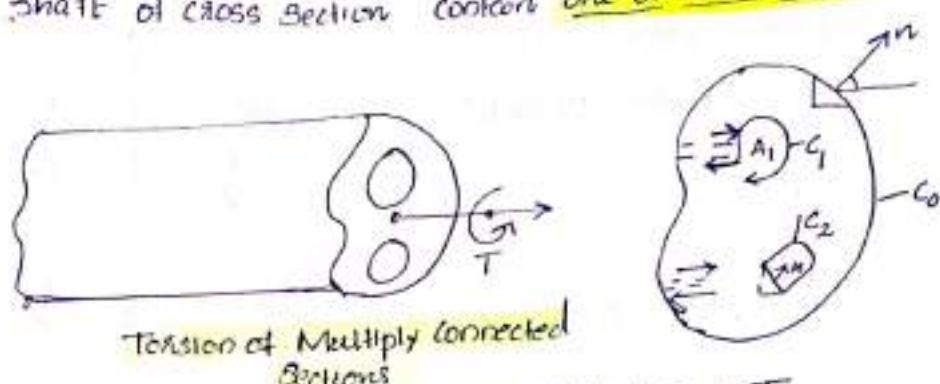
Shear stress on left, bottom, top leg of cell,  $\tau = \frac{\partial \phi}{\partial n} = \frac{\phi_1 - 0}{t} = \frac{\phi_1}{t}$

Shear stress on web;  $\tau = \frac{\partial \phi}{\partial n} = \frac{\phi_1 - \phi_2}{t} = 0$

Shear stress on the right, bottom and top leg of cell-2  $\tau = \frac{\partial \phi}{\partial n} = \frac{\phi_2 - 0}{t} = \frac{\phi_2}{t}$

# Torsion of Multiply connected Sections

- It is used for shafts of cross section contain one or more holes.



- Figure shows the section of shaft subjected to torque  $T$ .
  - $C_1$  and  $C_2$  are boundary of holes

Step 1 Similar to prandtl's approach. [ $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$ ] and  $\tau_{yz}$  and  $\tau_{zx}$  are non zero.

: Equilibrium eq  $\Rightarrow \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$

Hence  $\phi(x, y)$  = stress function

Note:  
 equilibrium eq - ①  $\frac{\partial \tau_{yz}}{\partial z} = 0$   
 equilibrium eq - ②  $\frac{\partial \tau_{zx}}{\partial z} = 0$   
 (Both satisfied)

Step 2 Stress components  $\tau_{zx} = \frac{\partial \phi}{\partial y}, \tau_{yz} = -\frac{\partial \phi}{\partial x}$

$$Y_{zx} = \frac{\tau_{zx}}{G_1} = \frac{1}{G_1} \frac{\partial \phi}{\partial y}, \quad Y_{yz} = \frac{\tau_{yz}}{G_1} = -\frac{1}{G_1} \frac{\partial \phi}{\partial x}$$

Step 3 Strain components Compatibility condition  $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = \text{constant}$

Step 4 Boundary Condition

$$n_x \frac{\partial \phi}{\partial y} - n_y \frac{\partial \phi}{\partial x} = 0 \Rightarrow \underbrace{\frac{\partial \phi}{\partial y} \frac{dy}{dx}}_{n_x} + \underbrace{\frac{\partial \phi}{\partial x} \frac{dx}{ds}}_{-n_y} = 0 \Rightarrow \frac{d\phi}{ds} = 0$$

Assume displacement

$$\text{in } z \text{ direction} \quad \psi_z = \Theta \psi(x, y)$$

$$\oint_{C_i} d\psi = \int_{C_i} \left( \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right) = 0$$

$$\text{we know } \frac{\partial \psi}{\partial x} = \frac{1}{G_1 \Theta} \tau_{zx} + y = \frac{1}{G_1 \Theta} \frac{\partial \phi}{\partial y} + 1$$

$$\frac{\partial \psi}{\partial y} = \frac{1}{G_1 \Theta} \tau_{yz} - x = \frac{1}{G_1 \Theta} \frac{\partial \phi}{\partial x} - x$$

$$\rightarrow \frac{1}{G_1 \Theta} \oint_{C_i} \left( \frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy \right) + \oint_{C_i} (y dx - x dy) = 0$$

Note  
 $\phi = 0$  (not assume for all boundaries, only assume one boundary)

$C_0$  = Hole boundary

$C_i$  = Boundary of hole

$\Theta_i$  = Corresponding angle of twist/unit length

$\oint_{C_i} \phi dy = \int_{GKH} y dx + \int_{HLI} y dx = \text{Area } GKGHH' - \text{Area } H'HLGKG$   
 $= \text{Area closed by } C_i - A_i$

$\oint_{C_i} \phi dx = \int_{LHK} x dy + \int_{KHL} x dy$   
 $= \text{Area } L'LGKGK' - \text{area } K'KGHL'$   
 $= -A_i$

$\therefore \oint_{C_i} (\phi dx - \phi dy) = 2A_i$

$\oint_{C_i} \left( \frac{\partial \phi}{\partial x} dx - \frac{\partial \phi}{\partial y} dy \right) = \oint_{C_i} \left( \frac{\partial \phi}{\partial y} \frac{dx}{dy} - \frac{\partial \phi}{\partial x} \frac{dy}{dx} \right) ds$   
 $= \oint_{C_i} \left( \frac{\partial \phi}{\partial y} \frac{dx}{dn} + \frac{\partial \phi}{\partial x} \frac{dy}{dn} \right) ds$  (From Fig.)  
 $= \oint_{C_i} \frac{\partial \phi}{\partial n} ds$   
 $\boxed{\oint_{C_i} \frac{\partial \phi}{\partial n} ds = 2GIO A_i}$  [Each boundary  $C_i, A_i$ : area enclosed by  $C_i$ ]

Step 7

Torque  $T = 2 \iint \phi dxdy$

Step 8

J-integral (Same as Saint-Venants approach)

$$\begin{aligned}
 J &= \iint_R (x^2 + y^2 + x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}) dxdy = \iint_R (x^2 + y^2 - \frac{1}{GIO} \frac{\partial \phi}{\partial x} - x^2 - \frac{1}{GIO} \frac{\partial \phi}{\partial y}) dxdy \\
 &= -\frac{1}{GIO} \iint_R \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) dxdy = \frac{1}{GIO} \iint_R \left( 2\phi - \frac{\partial}{\partial x}(x\phi) - \frac{\partial}{\partial y}(y\phi) \right) dxdy
 \end{aligned}$$

$$\boxed{J = \frac{2}{GIO} \iint_R \phi dxdy + \frac{1}{GIO} \oint_{C_i} \phi (ydx - xdy)}$$

using Gauss theorem Above eq  $\Rightarrow J = \frac{2}{GIO} \iint_R \phi dxdy + \frac{1}{GIO} \oint_{C_2} \phi (ydx - xdy) k_2 + \dots$   
 $C_i (i=0,1,2,3)$  become

using Saint Venants approach  $J = \frac{2}{GIO} \iint_R \phi dxdy + \frac{1}{GIO} (2k_1 A_1 + 2k_2 A_2 + \dots)$

$$J = \frac{2}{GIO} \left[ \iint_R \phi dxdy + \sum k_i A_i \right]$$

$$\boxed{\text{Torque} = GIO J = 2 \left[ \iint_R \phi dxdy + \sum k_i A_i \right]}$$

$A_i$  = Area enclosed by curve  $C_i$

① Derive the expression for hollow circular shaft?

Solution

$R_i$  = Inner radius

$R_o$  = outer radius

consider circular section of radius  $r$  from centre

Total twisting moment = Force  $\times$  Distance

$$T = \int_{R_i}^{R_o} dT = \int_{R_i}^{R_o} \frac{\tau}{R_o} 2\pi r^3 dr = \frac{\tau}{R_o} \times 2\pi \left[ \frac{\pi r^4}{4} \right]_{R_i}^{R_o} = \frac{6\pi \tau}{R_o} \left[ \frac{R_o^4 - R_i^4}{4} \right]$$
$$T = \frac{\pi \tau}{2} \left[ \frac{R_o^4 - R_i^4}{R_o} \right] \Rightarrow T = \frac{\pi}{16} \tau \left[ \frac{D_o^4 - D_i^4}{D_o} \right]$$

\* cohesive polar Moment of inertia  $I = \frac{\pi}{16} \left[ \frac{D_o^4 - D_i^4}{D_o} \right]$

$$\text{Angle of twist of shaft } \phi = \int_0^L \frac{T}{GJ} dz \Rightarrow \phi = \frac{TL}{GJ}$$

