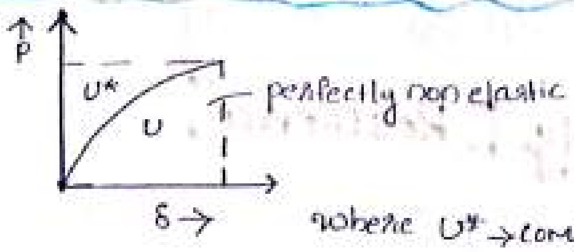
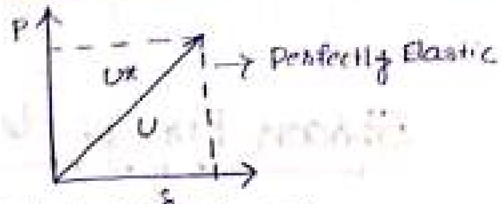
# Strain Energy of Deformation

**Strain Energy (u)**

When a load is applied on a body, the body is deformed, that is the work is done on the body. This work done is stored as some energy within body called Strain energy

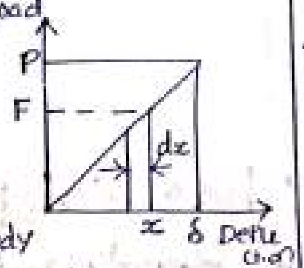



where U* → complementary Strain Energy

## Case-I : Strain Energy due to load P

Consider a solid body subjected to load P as shown in fig.


→ External force acts on body
→ Resisting force gradually develop
→ " increase from zero → Max value

From the graph $\frac{P}{\delta} = \frac{F}{x} \Rightarrow F = \frac{Px}{\delta}$

Strain Energy U = work done on the body

$$\Rightarrow U = \int_0^{\delta} F\,dx = \int_0^{\delta} \frac{Px}{\delta}\,dx = \frac{P}{\delta}\left[\frac{x^2}{2}\right]_0^{\delta}$$

$$\Rightarrow U = \frac{P}{\cancel{\delta}} \times \frac{\delta^{\cancel{2}}}{2} \Rightarrow \boxed{U = \frac{1}{2} P\delta} \text{ — (1)}$$

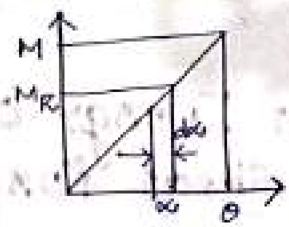
Strain Energy is half the product of force and deformation

$$U = \frac{1}{2} P\delta = \frac{1}{2}(\sigma A)(\epsilon l) = \frac{1}{2}(\sigma \epsilon V) = \frac{1}{2}\left(\sigma \times \frac{\sigma}{E} \times V\right)$$

$$\boxed{U = \frac{1}{2}\frac{\sigma^2 V}{E}} \text{ — (2)}$$

## Case II : Strain Energy due to moment

→ Consider cantilever beam subjected to external moment M at the free end.
→ Resisting moment $M_R$ varies with slope (α)





$$\frac{M}{\theta} = \frac{M_R}{\alpha}$$

$$M_R = \frac{M\alpha}{\theta}$$

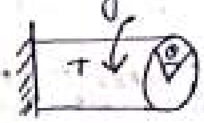
$$U = \int_0^{\theta} M_R\,d\alpha = \int_0^{\theta} \frac{M\alpha}{\theta}\,d\alpha = \frac{M}{\theta}\left[\frac{\alpha^2}{2}\right]_0^{\theta}$$

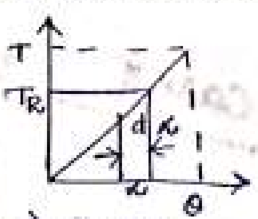
$$U = \frac{M}{\cancel{\theta} \times 2}\theta^{\cancel{2}} \Rightarrow \boxed{U = \frac{1}{2} M\theta}$$

Strain energy is half the product of moment and slope

## Case III : Strain Energy due to Torque T

Fig. shows shaft subjected to torque T
Resisting torque $T_R$ varies with twist (α) Angle.




From graph

$$\frac{T}{\theta} = \frac{T_R}{\alpha} \Rightarrow T_R = \frac{T\alpha}{\theta}$$

$$U = \int_0^{\theta} T_R\,d\alpha = \int_0^{\theta} \frac{T\alpha}{\theta}\,d\alpha = \frac{T}{\theta}\left[\frac{\alpha^2}{2}\right]_0^{\theta}$$

$$U = \frac{T}{2\cancel{\theta}}\theta^{\cancel{2}} \Rightarrow \boxed{U = \frac{1}{2} T\theta}$$

Strain Energy half product of Torque & Angle of twist

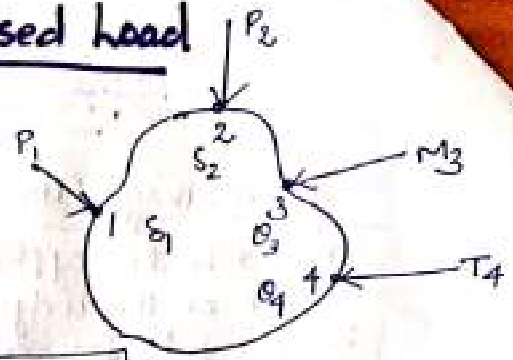
## Strain Energy due to Concentrated Load


⇒ we have concentrated load $P_1, P_2$ & $P_3$
⇒ $\delta_1, \delta_2, \delta_3$ corresponding displacement

$$\boxed{U = \frac{1}{2}(P_1\delta_1 + P_2\delta_2 + P_3\delta_3)}$$

Strain Energy of body is sum of strain Energy of individual Force.

## ⇒ Strain Energy due to Generalised Load




⇒ Here $P_1$ & $P_2$ are force acting at point 1 & 2

⇒ $M_3$ acting at 3 & $T_4$ acting at point 4

where $\delta_1, \delta_2$ = displacement due to $P_1$ & $P_2$

$\theta_3$ = slope due to $M_3$

$\theta_4$ = Angle of twist due to $T_4$

Principle of Superposition $\boxed{U = \frac{1}{2}\left[P_1\delta_1 + P_2\delta_2 + M_3\theta_3 + T_4\theta_4\right]}$

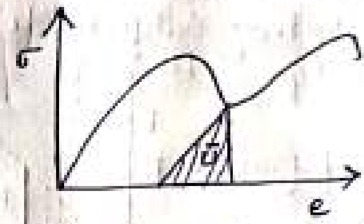
## Strain Energy density [Resilience]

It is defined as ratio of strain energy per unit Volume. The area of Stress strain graph is strain energy density ($\bar{U}$)

$$\bar{U} = \frac{\text{Strain Energy}}{\text{Volume}} = \frac{\frac{1}{2}P\delta}{V} = \frac{1}{2}\frac{P\delta}{AL} = \frac{1}{2}(\sigma\epsilon) \Rightarrow \boxed{\bar{U} = \frac{1}{2}\sigma\epsilon}$$

or

$$\bar{U} = \frac{\frac{1}{2}\frac{\sigma^2}{E}V}{V} \Rightarrow \boxed{\bar{U} = \frac{1}{2}\frac{\sigma^2}{E}}$$

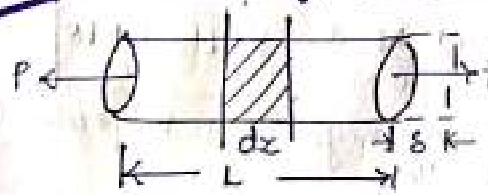



$\frac{P}{A} = \sigma$

$\frac{\delta}{L} = \epsilon$

## V.imp Strain Energy of bar Subjected to Axial Load, Shear force, Bending moment & Torque

### Case I Bar subjected to axial load / Axial force




⇒ Consider cylindrical bar subjected to an external load P, we have $U = \frac{1}{2}P\delta$

⇒ Replace δ by well known eq. $\delta = \frac{Pdx}{AE}$

$$U = \frac{1}{2}P\left(\frac{Pdx}{AE}\right) = \frac{1}{2}\frac{P^2dx}{AE}$$

Stress Tensor

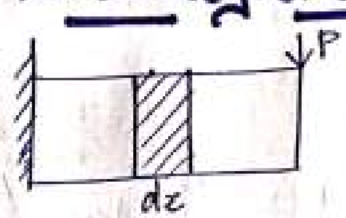
$$\begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Strain Tensor

$$\begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{U = \int_0^L \frac{1}{2}\frac{P^2dx}{AE}} \Rightarrow \boxed{U = \frac{P^2L}{2AE}}$$

### Case II Bar Subjected to Shear Load



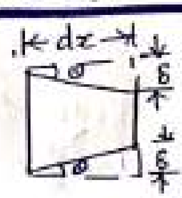




Fig. shows element block length dx of body subjected to Shear Stress

$$\tan\theta = \frac{\delta}{dx} \Rightarrow \theta = \frac{\delta}{dx} \Rightarrow \delta = \theta dx$$

we have $U = \frac{1}{2}P\delta = \frac{1}{2}P(\theta dx)$ —— ①

Shear Modulus $G = \frac{\text{Shear stress}}{\text{Shear strain}} \Rightarrow G = \frac{P/A}{\theta} \Rightarrow \boxed{\theta = \frac{P}{GA}}$

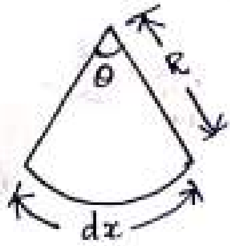
Sub: Value in eq—① $U = \frac{1}{2} P \left(\frac{P}{GA}\right) dx$

$\boxed{U = \int_0^L \frac{1}{2} \frac{P^2}{AG} dx}$ or Here Multiply A $\int_0^L \frac{1}{2} \frac{P^2 (A\,dx)}{A^2 G}$ → volume $\quad \frac{P^2}{A^2} = \tau^2$

or → $\boxed{U = \int_0^L \frac{\tau^2 dV}{2G}}$

## Case III Bar subjected to Moment (M)

Consider a beam subjected to moment M show in fig



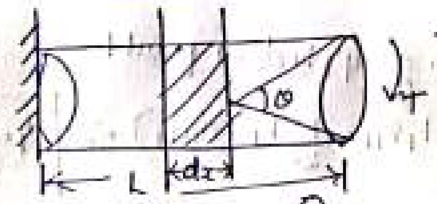

$\theta = \frac{dx}{R} \Rightarrow \frac{1}{R} = \frac{\theta}{dx} \rightarrow (a)$

We have $\frac{M}{I} = \frac{E}{R} \Rightarrow \frac{1}{R} = \frac{M}{EI} \rightarrow (b)$

equat (a) & (b) $\Rightarrow \frac{M}{EI} = \frac{\theta}{dx} \Rightarrow \theta = \frac{M\,dx}{EI}$

$U = \frac{1}{2} M\theta = \frac{1}{2} M \left(\frac{M\,dx}{EI}\right) \Rightarrow \boxed{U = \int_0^L \frac{1}{2} \frac{M^2 dx}{EI}}$

## Case IV Bar subjected to torque T




→ Here a shaft subjected to torque T

→ consider small element of distance dx

$U = \frac{1}{2} T\theta$ —— ①

we have $\frac{T}{J} = \frac{G\theta}{dx} \Rightarrow \theta = \frac{T\,dx}{JG}$

sub value of θ in eq—① $\Rightarrow U = \frac{1}{2} T \left(\frac{T\,dx}{GJ}\right)$

$\boxed{U = \int_0^L \frac{1}{2} \frac{T^2}{GJ} dx} \rightarrow \boxed{U = \frac{T^2 L}{2GJ}}$

### Uses of Strain Energy

① Used to find deflection of beam and frame by single load

② Design of members for dynamic load

③ Deflection of beam under impact load

### Proof Resilience

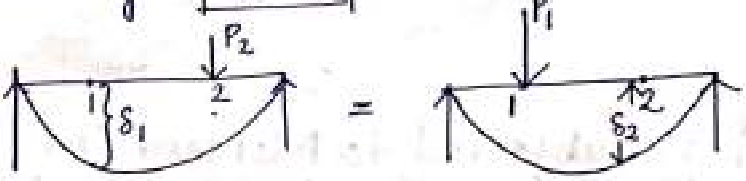
→ Maximum strain energy stored in the body up to elastic limit known as proof resilience

→ Modulus of Resilience → The proof resilience per unit volume called modulus of resilience

# Maxwell Reciprocal Relation

The relation states that the influence coefficient of displacement at point 1 due to Load at ② → point is equal to influence coefficient of displacement at point 2 due to Load at ① → point

Mathematically $\boxed{a_{12} = a_{21}}$




## proof

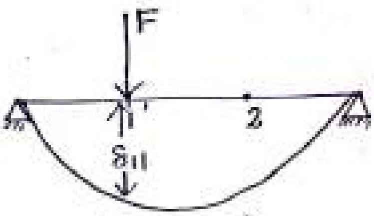



Figure (a)

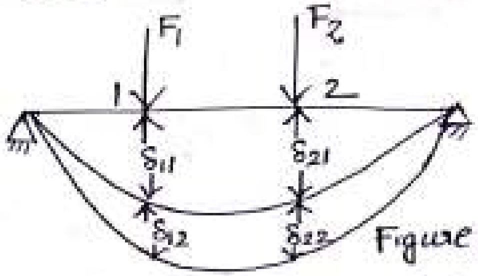



Figure (b)

→ consider simply supported beam carries two point loads $F_1$ & $F_2$ shown in fig

→ when a force $F_1$ is applied at any point 1. The deflection at point 1 and 2 are $\delta_{11}$ and $\delta_{21}$

→ when $F_2$ applied at point 2, the deflections at 1 & 2 are $\delta_{12}$ & $\delta_{22}$

Consider fig (a) → A force $F_1$ along applied at point 1

strain energy $U_1 = \frac{1}{2} F_1 \delta_{11} = \frac{1}{2} F_1 (a_{11} F_1) \Rightarrow \boxed{U_1 = \frac{1}{2} a_{11} F_1^2 \text{ --- } ①}$

Consider fig (b) →

Force $F_2$ applied at point 2 together with $F_1$. Therefore

Strain energy, $U_2 = \frac{1}{2} F_2 \delta_{22} + a_{12} F_1 F_2 = \frac{1}{2} F_2 (a_{22} F_2) + a_{12} F_1 F_2$

Total strain energy $= U_1 + U_2 = \frac{1}{2} a_{11} F_1^2 + \frac{1}{2} F_2^2 a_{22} + a_{12} F_1 F_2$ ——— ⓐ

Now repeat same step starting with load at 2 and then 1

total strain energy $= \frac{1}{2} a_{22} F_2^2 + \frac{1}{2} F_1^2 a_{11} + a_{21} F_1 F_2$ ——— ⓑ

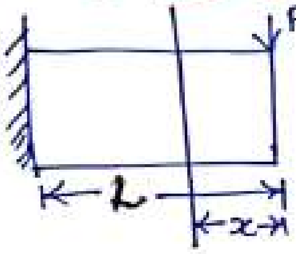
Elastic system strain energy only depend magnitude of applied load and independent of order in which applied

equate → eq ⓐ = eq ⓑ

$$\cancel{\frac{1}{2} a_{11} F_1^2} + \cancel{\frac{1}{2} F_2^2 a_{22}} + a_{12} \cancel{F_1 F_2} = \cancel{\frac{1}{2} a_{22} F_2^2} + \cancel{\frac{1}{2} F_1^2 a_{11}} + a_{21} \cancel{F_1 F_2}$$

$$\boxed{a_{12} = a_{21}}$$

① Find Strain Energy and transverse deflection at the point of application of load of cantilever.



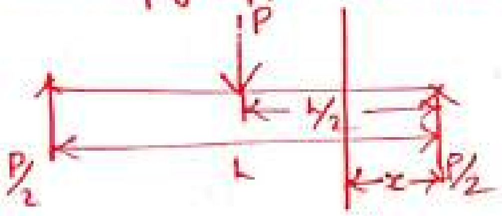

Solution

$$U = \frac{1}{2}\int_0^L \frac{M^2 dx}{EI} \quad \Big| \text{ where } M = Px$$

$$U = \frac{1}{2}\int_0^L \frac{P^2x^2}{EI}dx = \frac{P^2}{2EI}\left(\frac{x^3}{3}\right)_0^L = \underline{\underline{\frac{P^2L^3}{6EI}}}$$

Also $\frac{1}{2}P\delta = \frac{P^2L^3}{6EI} \Rightarrow \underline{\underline{\delta = \frac{PL^3}{3EI}}}$

② Find the transverse deflection at the point of application of load of Simply supported beam shown in fig.




Here $M = \frac{Px}{2}$

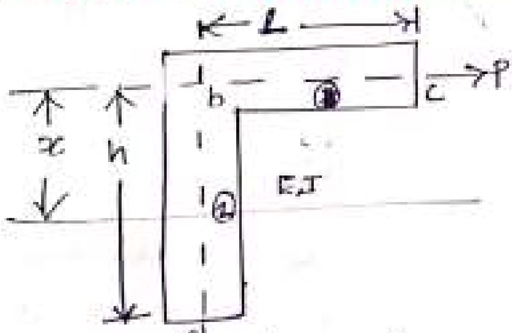
$$u = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{1}{2}\int_0^{L/2} \frac{\left(\frac{P}{2}x\right)^2}{EI}dx$$

$$u = \frac{1}{2}\frac{P^2}{4EI}\int_0^{L/2} x^2 dx = \frac{P^2}{8EI}\left(\frac{x^3}{3}\right)_0^{L/2} = \frac{P^2}{24EI}\frac{L^3}{8} = \underline{\underline{\frac{P^2L^3}{192EI}}}$$

Also $\frac{1}{2}P\delta = \frac{P^2L^3}{192EI} \Rightarrow \underline{\underline{\delta = \frac{PL^3}{96EI}}}$

HW ④ Find transverse deflection at the point of application of load of simply supported beam with UDL

③ Find the displacement along the direction of load P on bracket shown in fig.




Section ①

$$U_1 = \frac{1}{2}\int_0^L \frac{P^2 dx}{AE}$$

$$U_1 = \frac{P^2}{2AE}(x)_0^L \Rightarrow \underline{U_1 = \frac{P^2L}{2AE}}$$

Section ②

$(M = Px)$

$$u_2 = \int_0^h \frac{1}{2}\frac{M^2dx}{EI} = \frac{1}{2EI}\int_0^h (Px)^2 dx = \frac{P^2}{2EI}\left(\frac{x^3}{3}\right)_0^h = \underline{\frac{P^2h^3}{6EI}}$$

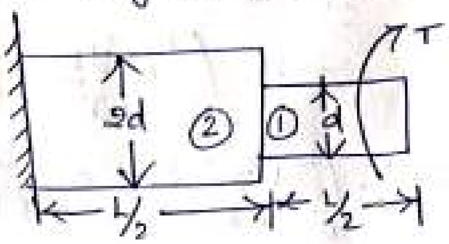
$$U_{total} = U_1 + U_2 = \underline{\frac{P^2L}{2AE} + \frac{P^2h^3}{6EI}}$$

Also $\frac{1}{2}P\delta = \frac{P^2L}{2AE} + \frac{P^2h^3}{6EI}$

$$\frac{1}{\cancel{2}}\cancel{P}\delta = \cancel{P}\left(\frac{PL}{\cancel{2}AE} + \frac{Ph^3}{\cancel{6}_3EI}\right)$$

$$\Rightarrow \boxed{\delta = \frac{PL}{AE} + \frac{Ph^3}{3EI}}$$

④ A circular shaft consist of 2 portion AB & BC having same length and cross section.
Determine angle of twist when shaft subjected to torque T




Section①

$$U_1 = \frac{1}{2}\int_0^{L/2} \frac{T^2 dx}{GJ} = \frac{1}{2}\frac{T^2}{GJ}(x)_0^{L/2}$$

$$U_1 = \frac{T^2}{2GJ}\left(\frac{L}{2}\right) = \underline{\underline{\frac{T^2L}{4GJ}}}$$

Section②

$$U_2 = \frac{1}{2}\int_0^{L/2} \frac{T^2 dx}{GJ} = \frac{1}{2}\frac{T^2}{GJ}(x)_0^{L/2} = \frac{T^2L}{4GJ}$$

where $J = \frac{\pi}{32}d^4$

$J = \frac{\pi}{32}(2d)^4 = \frac{\pi}{32}16d^4$

$\boxed{J = 16J}$

$$U_2 = \frac{T^2L}{4G(16J)} = \underline{\underline{\frac{T^2L}{64GJ}}}$$

$$U_{total} = U_1 + U_2 = \frac{T^2L}{4GJ} + \frac{T^2L}{64GJ} = \frac{T^2L}{4GJ}\left(1+\frac{1}{16}\right) = \frac{17T^2L}{4\times16GJ}$$

$U_{total}$ Also $\Rightarrow$ $\frac{1}{\cancel{2}}T\theta = \frac{17T^{\cancel{2}}L}{4\times\cancel{16}_{8}GJ}$ $\Rightarrow$ $\boxed{\theta = \frac{17TL}{32GJ}}$

⑥ A tensile load of 80kN applied suddenly to a circular bar of 50mm dia & 4m long. Determine

(a) Maximum instantaneous stress induced $\quad E = 2\times10^5 N/mm^2$

(b) Instantaneous elongation in the bar

(c) Strain Energy absorbed in the bar

Solution

$d = 50mm$, $A = \frac{\pi}{4}d^2 = \frac{\pi}{4}50^2 = \underline{1962.5mm^2}$, $L = 4M = 4000mM$

$V = Al = 1962.5\times4000 = \underline{7850000 mm^3}$, $\underline{P = 80\times10^3 N}$

(a) Max. instantaneous stress $\Rightarrow \sigma = \frac{2P}{A} = \frac{2\times80\times10^3}{1962.5} = \underline{81.53 N/mm^2}$

(b) Elongation bar $\delta = \frac{\sigma}{E}l = \frac{81.53}{2\times10^5}\times4000 = 1.63mM$

(c) Strain Energy $U = \frac{1}{2}\frac{\sigma^2}{E}\times V = \frac{1}{2}\frac{\times81.53^2}{2\times10^5}\times7850000 = \underline{\underline{130450 NmM}}$

⑦ For a given stress tensor at a point on the steel object with $E = 207\times10^6 kpa$ & $G = 80\times10^6 kpa$. Determine value of strain energy density.

2019 DEC

$$[\sigma] = \begin{bmatrix} 9 & 0 & 3 \\ 0 & -10 & 1 \\ 3 & 1 & 112 \end{bmatrix}\times10^3 kpa$$

Solution

Strain Energy density $= \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})$

where $(\sigma_x = 9, \sigma_y = -10, \sigma_z = 112, \tau_{xy} = 0, \tau_{xz} = 3, \tau_{yz} = 1)\times10^3 kpa$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y+\sigma_z)\right) = \frac{1}{207\times10^{6}}\left(9-0.294(-10+112)\right)10^{3}$$
$$= -1.01\times10^{4}$$
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x+\sigma_z)\right) = \frac{1}{207\times10^{6}}\left(-10-0.294(9+112)\right)10^{3}$$
$$= 2.656\times10^{4}$$
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x+\sigma_y)\right) = \frac{1}{207\times10^{6}}\left(112-0.294(9-10)\right)10^{3} = \underline{5.4\times10^{4}}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0,\quad \gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{3\times10^{3}}{80\times10^{6}} = \underline{0.375\times10^{+4}}$$
$$\gamma_{yz} = \tau_{yz}/G = \frac{1\times10^{3}}{8\times10^{6}} = \underline{1.25\times10^{+4}}$$

$$G = \frac{E}{2(1+\nu)}$$
$$80\times10^{6} = \frac{207\times10^{6}}{2(1+\nu)}$$
$$160(1+\nu) = 207$$
$$160+160\nu = 207$$
$$160\nu = 47$$
$$\underline{\nu = 0.29375}$$

Strain Energy density $= \frac{1}{2}\left(9\times-1.01 + -10\times2.656 + 112\times5.4 + 0 + 0.375\times3 + 1\times1.25\right)10^{3}\times10^{4}$

$= \underline{2857\times10^{6}\,kpa} = \underline{2857\,Gpa}$

(8) For given tensorial ~~on~~ steel object with $E = 207\times10^{6}$ kpa & $G = 80\times10^{6}$ kpa. Determine Strain energy density?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3/2 \\ -1 & 3/2 & 0 \end{bmatrix}\times10^{-3}$$

Solution

Strain energy density $= \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)$

$\sigma_x = 2\mu\varepsilon_{xx} + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = (2\times80\times1 + 114.15(1+(-3)+0))10^{6}10^{-3}$
$= -68.3\times10^{3}$ kpa

$\sigma_y = 2\mu\varepsilon_y + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 2\times80\times(-3) + 114.15(1-3+0)10^{6}10^{-3}$
$= -708.3\times10^{3}$ kpa

$\sigma_z = 2\mu\varepsilon_z + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 2\times80\times0 + 114.15(1-3+0)\,10^{6}10^{-3}$
$= -228.30\times10^{3}$ kpa

$\tau_{xy} = \tau_{yx} = \mu\gamma_{xy} = 0$, $\tau_{xz} = \tau_{zx} = \mu\gamma_{xz} = 80\times10^{6}\times-2\times10^{-3}$
$= -160\times10^{3}$ kpa

$\tau_{yz} = \tau_{zy} = \mu\gamma_{yz} = 80\times10^{6}\times3\times10^{-3} = 240\times10^{3}$ kpa

Strain Energy $= \frac{1}{2}\left(-68.3\times1 + (-708.3\times-3) + 0 + 0 + -160\times2 + 240\times3/2\right)$

$= \underline{3096.6\,kpa}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3/2 \\ -1 & 3/2 & 0 \end{bmatrix}\times10^{-3} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix}$$

$E = 207\times10^{6}$ kpa

$G = \frac{E}{2(1+\nu)} = \mu$

$80\times10^{6} = \frac{207\times10^{6}}{2(1+\nu)} \Rightarrow \nu = 0.294$

$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{0.294\times207\times10^{6}}{(1+.294)(1-2\times.294)}$

$\lambda = \underline{114.15\times10^{6}\,kpa}$

(9) Calculate downward displacement at the point of application of load of Structure shown in fig. All members are axial force members having same cross section area $5cm^2$. Member 1 to 4 have same length of 0.5m. $E = 200$ Gpa.

Solution

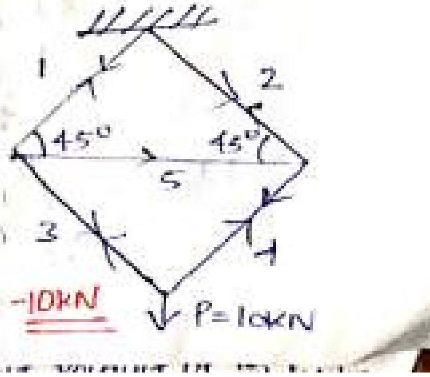
$l_1 = l_2 = l_3 = l_4 = 0.5$m $\quad \boxed{l_5 = 0.5\sqrt{2} = 0.707\,m}$

$A_1 = A_2 = A_3 = A_4 = A_5 = 5cm^2 = \underline{5\times10^{-4}m^2}$

Method of Joint/Section

$F_1 = F_2 = F_3 = F_4 = 10\times\sqrt{2} = \underline{7.071N}$

$F_5 = ? \Rightarrow \cos 45 = \frac{10\times\sqrt{2}}{F_5} \Rightarrow F_5 = -P = \underline{-10kN}$



$$U = 4\left[\frac{F_1^2 l_1}{2EA_1}\right] + \frac{F_5^2 L_5}{2EA_5} = 4\times\left[\frac{7.0711^2\times10^6\times0.5}{2\times200\times10^9\times5\times10^{-4}} + \frac{10^2\times10^6\times0.707}{2\times200\times10^9\times5\times10^{-4}}\right.$$

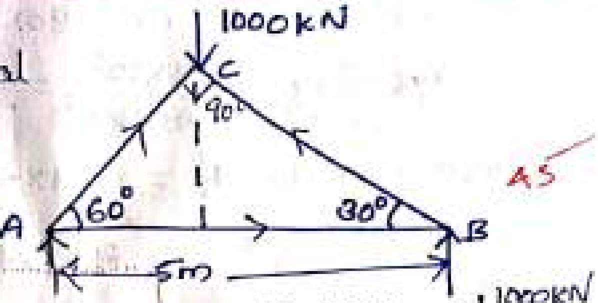
$$= \underline{0.8535]}$$

$$\frac{1}{2}P\delta = 0.8535 \Rightarrow \delta = \frac{0.8535}{10\times10^3} = \underline{0.1707mm}$$

(10) Find the downward displacement at the point of load applied for the given figure? Crosssection of members $2cm^2$ & $E = 200Gpa$

2019 Dec

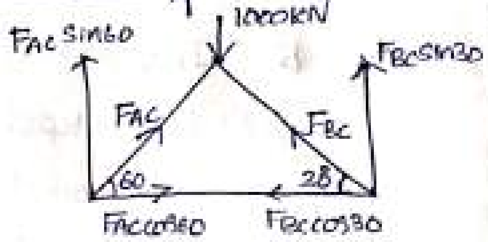
Solution




Cable made of linear elastic material complementary energy equal to strain energy.

$$U^* = U = \frac{1}{2}\frac{P^2L}{AE}$$

$$U^* = U = \frac{1}{2}\frac{F_{AC}^2 l_{AC}}{AE} + \frac{1}{2}\frac{F_{BC}^2 l_{BC}}{AE} + -$$




where $\cos60 = \frac{AC}{5} \Rightarrow AC = 2.5$

$\cos30 = \frac{BC}{5} \Rightarrow BC = 4.33$

$$U = \frac{1}{2}\frac{866^2\times10^6\times2.5}{2\times10^{-4}\times200\times10^9} + \frac{1}{2}\frac{500^2\times10^6\times4.33}{2\times10^{-4}\times200\times10^9}$$

$$U = 23436.125 + 13531.25 = 36967.375]$$

$$\frac{1}{2}P\delta = U \Rightarrow \delta = \frac{36967.375\times2}{1000\times10^3} = \underline{0.0739m}$$

$\Sigma F_x = 0$

$\Rightarrow F_{AC}\cos60 - F_{BC}\cos30 + \cancel{F_{AB}} = 0$ — (1)

$\Sigma F_y = 0$

$F_{AC}\sin60 + F_{BC}\sin30 - 1000 = 0$ — (2)

$0.5F_{AC} - 0.866F_{BC} + \cancel{F_{AB}} = 0$ — (1)

$0.866F_{AC} + 0.5F_{BC} = 1000$ — (2)

eq (1) $F_{AC} = 1.732F_{BC}$

Sub: (2)

$0.866\times1.732F_{BC} + 0.5F_{BC} = 1000$

$1.4999F_{BC} + 0.5F_{BC} = 1000$

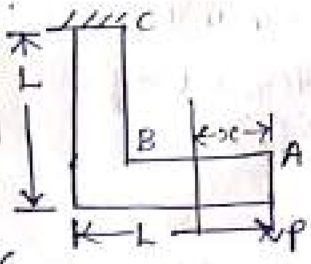
$F_{BC} = \underline{500\ kN}$

$F_{AC} = 866\ kN$

(11) Determine deflection at point A due to applied Load P. The flexural rigidity of frame EI Effect of axial load neglected.

2019 Dec

Solution




$$U_{total} = U_{AB} + U_{BC}$$

$$U_{AB} = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{1}{2}\int_0^L \frac{(Px)^2}{EI} \qquad \Big|\ M = -Px$$

$$= \frac{P^2}{2EI}\left(\frac{x^3}{3}\right)_0^L = \underline{\frac{P^2L^3}{6EI}}$$

$$U_{BC} = \frac{1}{2}\int_0^L \frac{M^2}{EI}dy \qquad \Big|\ M = PL$$

$$= \frac{P^2L^2}{2EI}(y)_0^L \Rightarrow \frac{P^2L^3}{2EI}$$

$$E = E_{AB} + E_{BC}$$

$$U = U_{AB} + U_{BC}$$

$$U = \frac{P^2L^3}{6EI} + \frac{P^2L^3}{2EI}$$

$$U = \frac{8P^2L^3}{12EI}$$

$$\frac{1}{2}P\delta = \frac{8P^2L^3}{12EI} \Rightarrow \underline{\underline{\delta = \frac{8PL^3}{6EI}}}$$

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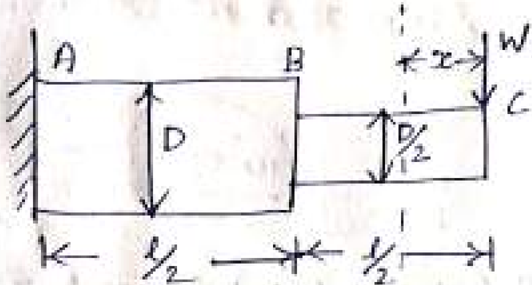
12 A cylindrical cantilever rod ABC of length $l$ carries a load $W$ at the free end. Find deflection at free end.




$I_{AB} = \frac{\pi}{64} D^4$

$I_{BC} = \frac{\pi}{64}\left(\frac{D}{2}\right)^4 = \frac{\pi D^4}{1024} = I$

$I_{AB} = 16 \times \frac{\pi D^4}{1024} = 16I$

$M = Wx$

$$U_{BC} = \int_0^{l/2} \frac{M^2\,dx}{2EI} = \int_0^{l/2} \frac{(Wx)^2}{2EI}\,dx = \frac{W^2}{2EI}\left(\frac{x^3}{3}\right)_0^{l/2} = \frac{W^2 l^3}{48EI}$$

$$U_{AB} = \int_{l/2}^{l} \frac{M^2\,dx}{2EI} = \int_{l/2}^{l} \frac{(Wx)^2}{2E(16I)}\,dx = \frac{W^2}{32EI}\left(\frac{x^3}{3}\right)_{l/2}^{l}$$

$$= \frac{W^2}{32EI \times 3}\left[l^3 - \frac{l^3}{8}\right] = \frac{7}{768}\frac{W^2 l^3}{EI}$$

$$U_{total} = U_{BC} + U_{AB} = \frac{W^2 l^3}{48EI} + \frac{7}{768}\frac{W^2 l^3}{EI}$$

Also $\frac{1}{\cancel{2}} \cancel{W}\delta = \frac{W^{\cancel{2}} l^3}{\cancel{48}_{24} EI} + \frac{7}{\cancel{768}_{384}}\frac{W^{\cancel{2}} l^3}{EI} \Rightarrow \boxed{\delta = \frac{Wl^3}{24EI} + \frac{7Wl^3}{384EI}}$

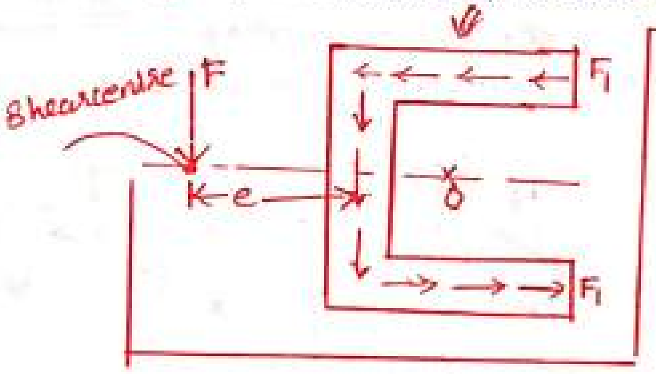
# Module-4

## SHEAR CENTRE FOR THIN WALLED SECTIONS

Statement: When a beam has unsymmetrical crosssection if the applied Load passes through centroid of crosssection of beam, there will be twisting in addition to bending. Inorder to avoid twisting and allow pure bending the load has to be applied through some other appropriate point. This point is called Shear centre or centre of twist

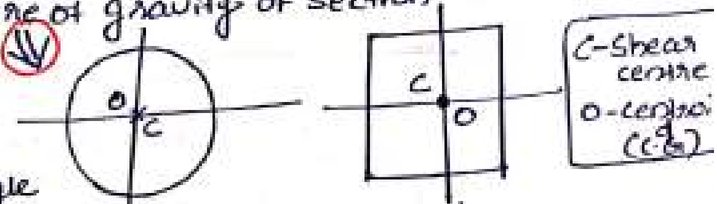
OR

Shear centre is a point where the applied Load is balanced by set of Shear forces, to allow bending and to prevent ~~bending~~ twisting.

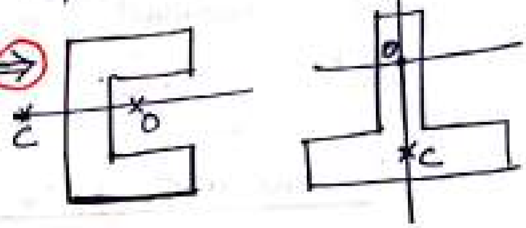



### Properties of shear centre

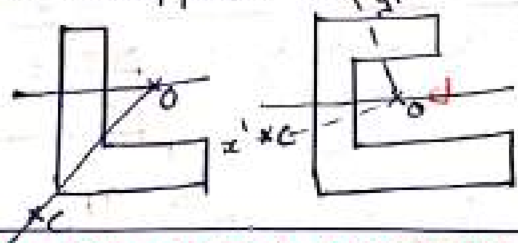
① If crosssectional area has two axes of symmetry, the shear centre concide with centre of gravity of section.




② If the crosssectional area has single axis of symmetry, the shear centre lies on the axis of symmetry itself

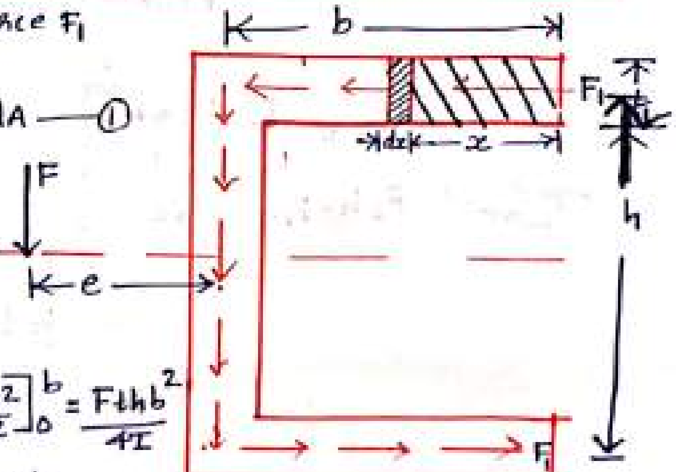



③ When section is unsymmetrical, Shear centre lies opposite side of Open part




### DETERMINATION OF SHEAR CENTRE FOR A CHANNEL SECTION

Consider a channel section with Shear force $F_1$ and applied force F as shown in fig.




Shear stress on the section $\tau = \frac{F_1}{dA} \Rightarrow F_1 = \tau dA$ —①

we know $\tau = \frac{FA\bar{y}}{Ib}$ and $dA = dx t$ Sub eq—①

$$F_1 = \frac{F(xt) \times \frac{h}{2}}{It} \times dx t$$

$A = xt$
$\bar{y} = h/2$
$b = t$

For entire flange $F_1 = \frac{Fth}{2I}\int_0^b x\,dx = \frac{Fth}{2I}\left[\frac{x^2}{2}\right]_0^b = \frac{Fthb^2}{4I}$

Inorder to prevent twisting, equate the moment

$F \times e = F_1 \times h \Rightarrow e = \frac{F_1 h}{F} \Rightarrow e = \frac{Fthb^2 \times h}{F \times 4I} \Rightarrow \boxed{e = \frac{b^2h^2t}{4I}}$

where b = distance from end of flange to centroid of web

I - Total Moment of inertia
t = Thickness of web/flange
h = ⊥lar distance b/w centroid of flanges

① Locate the shear centre for a channel section 100×100×10mm

Solution

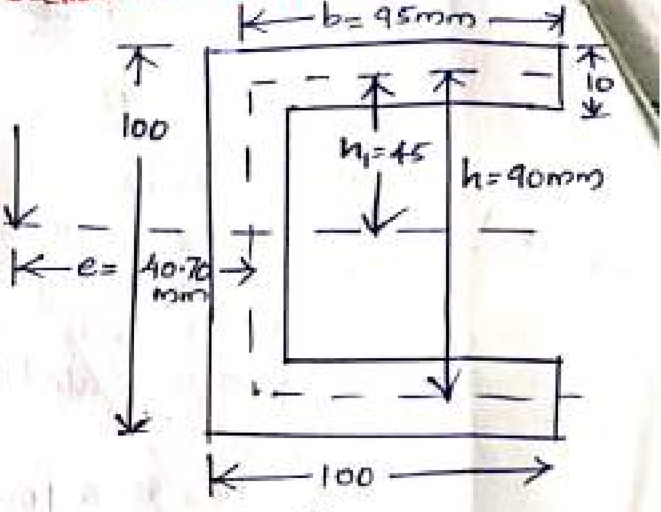
Shear centre $e = \frac{b^2h^2t}{4I}$ ——①

When b = 95mm, h = 90mm, t = 10mm

$I = I_{web} + 2(I_{flange} + Ah_1^2)$

$I = \frac{10 \times 80^3}{12} + 2\left(\frac{100 \times 10^3}{12} + 100 \times 10 \times 45^2\right)$

$= 4.49 \times 10^6 \, mm^4$




Sub: Value of b, h, t & I in eq-① ⟹ $e = \frac{b^2h^2t}{4I} = \frac{95^2 \times 90^2 \times 10}{4 \times 4.49 \times 10^6} = \underline{40.70mm}$

② Determine the position of shear centre for a channel section of 450×650×1...

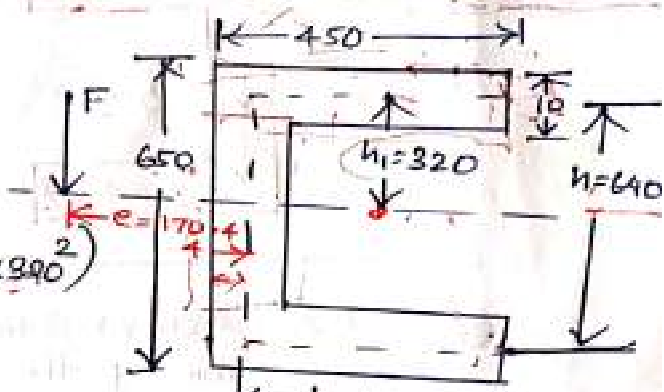
Shear centre $e = \frac{b^2h^2t}{4I}$ ——①

where b = 445mm, h = 640mm, t = 10mm

$I = I_{web} + 2(I_{flange} + Ah_1^2)$

$I = \frac{10 \times 630^2}{12} + 2\left(\frac{450 \times 10^3}{12} + 450 \times 10 \times 320^2\right)$

$I = 1.13 \times 10^9 \, mm^4$




Sub: b, h, t & I eq① ⟹ $e = \frac{b^2h^2t}{4I} = \frac{445^2 \times 640^2 \times 10}{4 \times 1.13 \times 10^9} = \underline{179.44mm}$

## DETERMINATION OF SHEAR CENTRE FOR I-SECTION

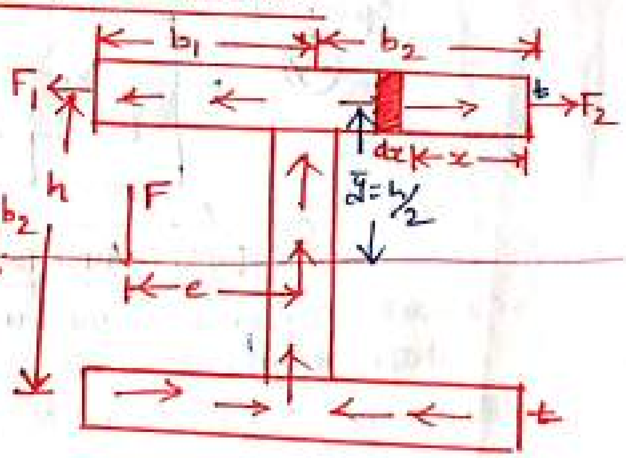
where $F_2 = \tau dA$ ⟹ | $\tau = \frac{FA\bar{y}}{Ib} = \frac{F(x t)h/2}{It}$, $dA = dx\,t$

$F_2 = \frac{F(xt)h/2}{It} dx\,t$

Entire flange $F_2 = \frac{Fth}{2I}\int_0^{b_2} x\,dx = \frac{Fth}{2I}\left(\frac{x^2}{2}\right)_0^{b_2}$

$$\boxed{F_2 = \frac{Fthb_2^2}{4I}}$$

Similarly $\boxed{F_1 = \frac{Fthb_1^2}{4I}}$




equate moment $F_2h - F_1h = F \times e \Rightarrow e = \frac{(F_2 - F_1)h}{F} = \left(\frac{\cancel{F}thb_2^2}{4I} - \frac{\cancel{F}thb_1^2}{4I}\right)\frac{h}{\cancel{F}}$

$$\boxed{e = \frac{h^2t(b_2^2 - b_1^2)}{4I}}$$

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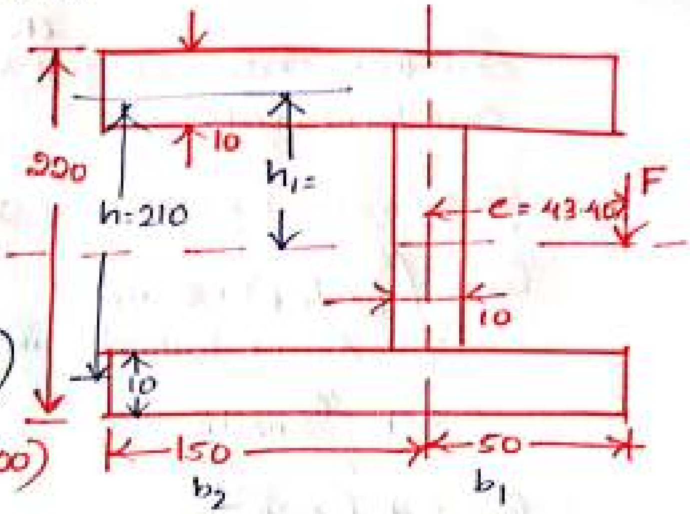
... shear centre for given I-section

Shear centre $e = \frac{h^2 t (b_2^2 - b_1^2)}{4I}$




where h=210mm, t=10mm

$b_2$=150mm, $b_1$=50mm

$I = I_{web} + 2(I_{flange} + Ah^2)$

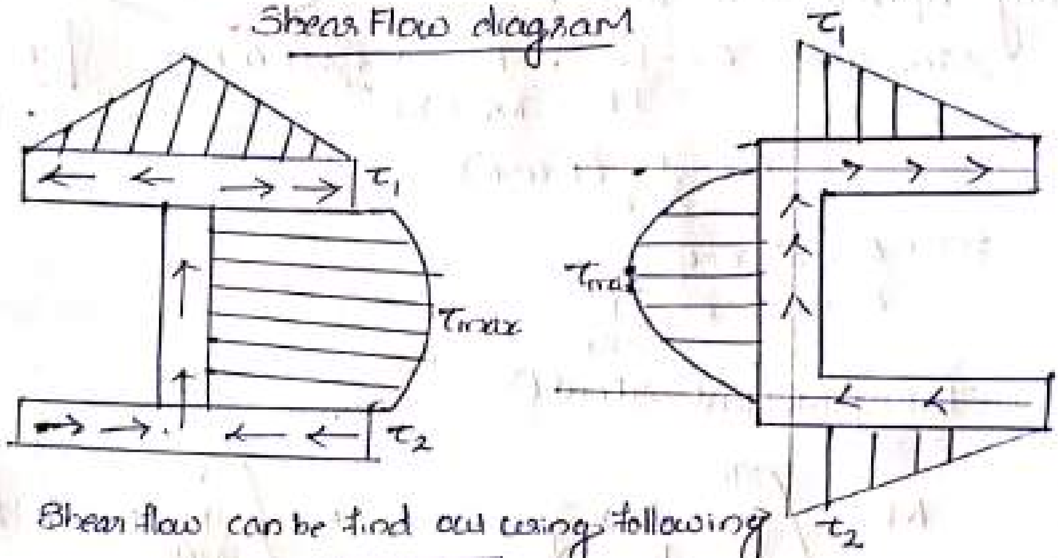
$= \frac{10 \times 200^3}{12} + 2\left(\frac{200 \times 10^3}{12} + (200 \times 10) 105^2\right)$

$= 6666666.67 + 2(16666.67 + 22050000)$

$= 0.508 \times 10^8 mm^4$

Shear centre $e = \frac{h^2 t (b_2^2 - b_1^2)}{4I} = \frac{210^2 \times 10 (150^2 - 50^2)}{4 \times 508 \times 10^6} = 43.40 mm$

Shear Flow diagram



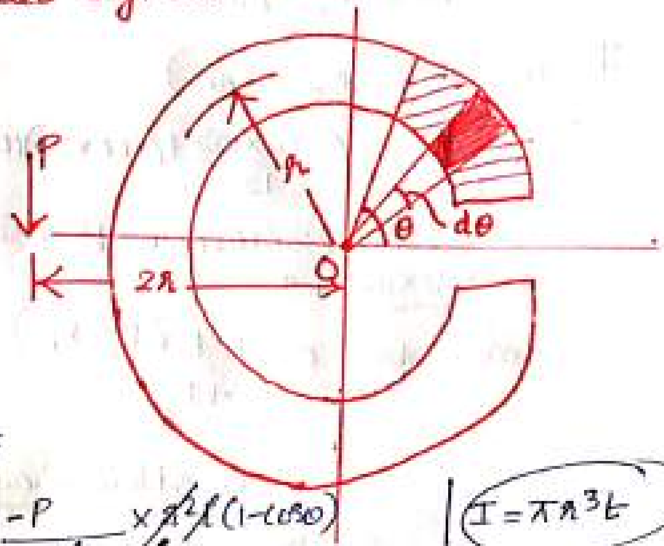

Shear flow can be find out using following relation $\boxed{q = \frac{FA\bar{y}}{I}}$

Determine shear centre for a circular open section under bending

OR

Show that shear centre for a circular open ring should be 2r dist ance from centre

Static moment of area of shaded segment




$$Q = \int_0^{\theta} (r\,d\theta\,t) \times r\sin\theta$$

Area x ⊥ular distance

$$= r^2 t \int_0^{\theta} \sin\theta\, d\theta$$

$$Q = r^2 t\,[-\cos\theta]_0^{\theta}$$

$$\boxed{Q = r^2 t\,[1-\cos\theta]}$$

Vertically upward shear force F

Shear stress $\tau = \frac{-PQ}{It} = \frac{-P}{\pi r^{\cancel{3}} t \times \cancel{t}} \times \cancel{r^2}\cancel{t}(1-\cos\theta)$

$I = \pi r^3 t$

↳ M.I of circular section

$$\tau = \frac{-P}{\pi r t}(1-\cos\theta)$$

when $\theta = 0 \Rightarrow \tau = 0$

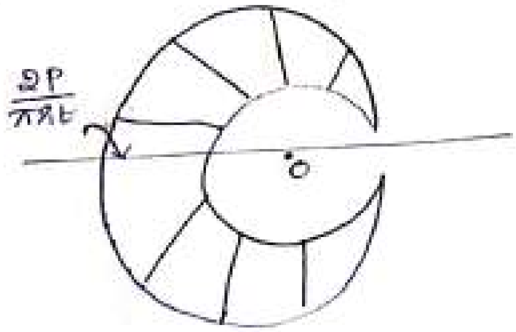
$\theta = 180 \Rightarrow \tau = \frac{-2P}{\pi r t}$

Taking moment about O

$$M = \int_0^{2\pi} \underbrace{\tau}_{\text{stress}} \times \underbrace{r\,d\theta}_{\text{Area}} \times \underbrace{r}_{\text{⊥ular distance}} = \int_0^{2\pi} \frac{-P}{\pi \cancel{r} t}(1-\cos\theta)\, r\,d\theta \times \cancel{r}$$

$$M = \frac{-Pr}{\pi}\int_0^{2\pi}(1-\cos\theta)\,d\theta = \frac{-Pr}{\pi}\,[\theta - \sin\theta]_0^{2\pi} = \frac{-Pr}{\cancel{\pi}}(2\cancel{\pi})$$

$$\boxed{M = -2Pr}$$




Force P must be applied 2r distance from centre of section to avoiding twisting

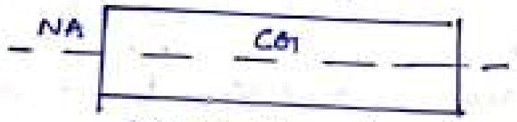
M = Force x distance

$M = -P \times (2r)$

$\therefore \boxed{e = 2r}$

# CURVED BEAMS

- A beam in which the neutral axis in unloaded condition is curved instead of straight is called a curved beam
- If a beam is orginally curved before applying the bending moment such beams are coming under curved beam
  Eg: Clamp, hook, ring
- In this beam the line of action of load does not passes through centroid of section.

| Straight Beams | Curved beams |
|---|---|
| • Neutral axis of cross section passes through centroid of section. • Variation of bending stress in lines  Eg: cantilever beam | * Does not coincide with centroidal section, it is shifted towards of curvature of beam. * Distribution of stress hyperbolic  Eg: U-clamp, rings |

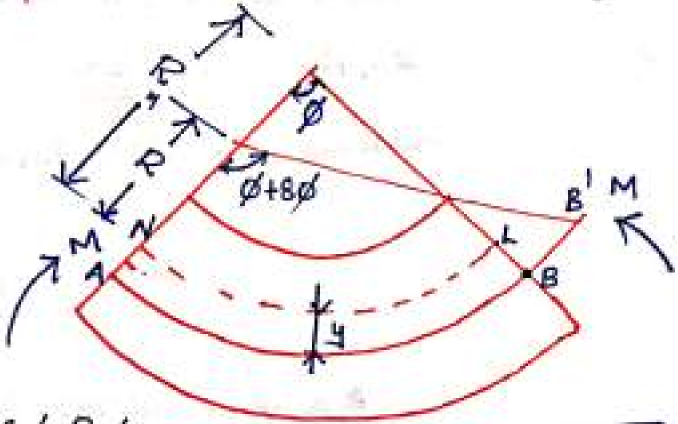




* N.A lies b/w centroidal axis & centre of curvature

## WINKLER-BATCH THEORY [STRESSES IN CURVED BEAMS]

Consider small length of a bar





Strain of fiber $\boxed{\varepsilon = \frac{AB' - AB}{AB}}$ —— ①

where $AB' = (R+y)(\phi+\delta\phi)$ and $AB = (R_n+y)\phi$

Sub eq-① $\varepsilon = \frac{(R+y)(\phi+\delta\phi) - (R_n+y)\phi}{(R_n+y)\phi}$

$$\varepsilon = \frac{R(\phi+\delta\phi) + \cancel{y\phi} + y\delta\phi - R_n\phi - \cancel{y\phi}}{(R_n+y)\phi} = \frac{R_n\phi + y\delta\phi - R_n\phi}{(R_n+y)\phi} = \frac{y}{(R_n+y)}\frac{\delta\phi}{\phi}$$

$\boxed{R_n\phi = R(\phi+\delta\phi)}$

$E = \frac{\sigma}{\varepsilon}$

$$\frac{\delta\phi}{\phi} = \frac{R_n+y}{y} \times \varepsilon = \frac{R_n+y}{y} \times \frac{\sigma}{E} \Rightarrow \frac{\delta\phi}{\phi} = \frac{R_n+y}{y} \times \frac{\sigma}{E} \text{ —— ①}$$

Moment $M = \int_A \sigma y \, dA \Rightarrow \boxed{M = E\frac{\partial\phi}{\phi} \times Ae}$ —— ②

Sub: eq-① in eq-②

$$M = \cancel{E}\left(\frac{R_n+y}{y}\right)\frac{\sigma}{\cancel{E}} Ae = \sigma \frac{R_n+y}{y} Ae$$

Bending stress $\boxed{\sigma = \frac{M}{Ae}\frac{y}{R_n+y} \quad \text{or} \quad \sigma = \frac{-M}{Ae}\frac{y}{R_n-y}}$

Above eq known as Winkler-Batch Equation.

Assumptions

1. Material of beam homogenous & isotropic
2. Material of beam obey hooks law
3. plane before bending remain plane after bending
4. cross section has axis of symmetry in plane along length of beam
5. Youngs Modulus same in both tension & compression

# Location of Neutral Axis of Rectangular Section

Stress due to bending

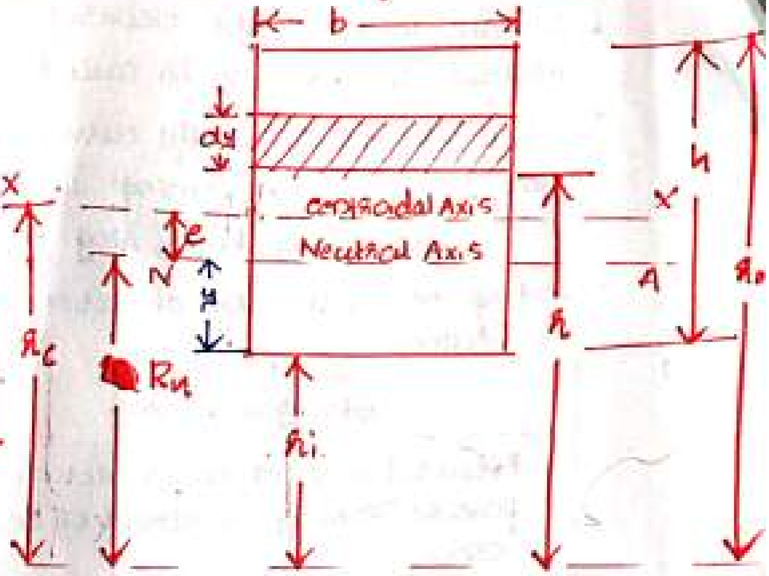
$$\sigma = \pm \frac{M}{Ae}\left[\frac{y}{R_n - y}\right]$$

+ ⇒ Tension
− ⇒ Compression

Imp

$$\boxed{R_n = \frac{h}{\ln\left(\frac{r_o}{r_i}\right)} = \frac{h}{\ln\left(\frac{r_c + h/2}{r_c - h/2}\right)}}$$

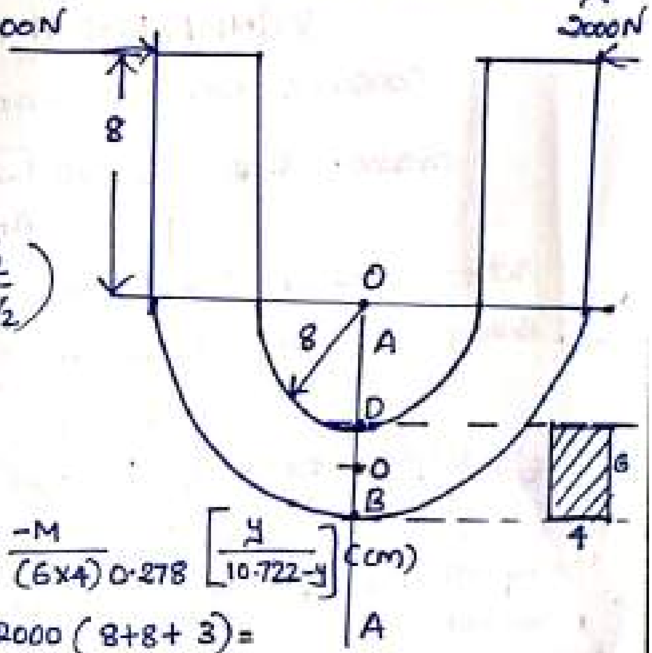
$$\boxed{\begin{aligned} e &= r_c - R_n \\ r_c &= r_i + h/2 \end{aligned}}$$




where M = B.M due to above centroidal axis
$r_i$ = radius of inner fibre
$r_o$ = radius of outer fibre
$r_c$ = radius of centroidal axis
$R_n$ = radius of Neutral axis

e = distance b/w NA & CG [Eccentricity]
y = distance b/w NA & outer or inner fibre
b = Breadth of rectangular section
h = depth of rectangular section

(1) Determine the maximum tensile stress at a section A-A shown in fig




For section $h = 6cm$, $b = 4cm$

$r_c = r_i + h/2 = 8 + \frac{6}{3} = 11cm$

Radius of Neutral Axis $R_n = \frac{h}{\ln\left(\frac{r_c + h/2}{r_c - h/2}\right)}$

$R_n = \frac{6}{\ln\left(\frac{11 + 6/2}{11 - 6/2}\right)} = 10.722cm$

$e = r_c - R_n = 11 - 10.722 = 0.278$

Bending stress $\sigma = \frac{-M}{Ae}\left[\frac{y}{R_n - y}\right] = \frac{-M}{(6 \times 4)\,0.278}\left[\frac{y}{10.722 - y}\right]$

Bending Moment section A-A [B.M at O] $\Rightarrow M = 2000\,(8 + 8 + 3) = 38000$ N-cm

$\sigma_D = \frac{-38000}{24 \times 0.28}\left[\frac{2.72}{10.722 - 2.72}\right] = -1922.61 N/cm^2$

$y_D = 3 - 0.28 = 2.72$

$\sigma_B = \frac{-38000}{24 \times 0.28}\left[\frac{-3.28}{10.722 - (-3.28)}\right] = 1324.8 N/cm^2$

$y_c = 6 - 2.72 = 3.28$

(−) towards outer

$y_c = -3.28$

Direct Stress $= P/A = \frac{-2000}{24} = -83.33 N/cm^2$ (−ve compressive)

Total stress at B = 1324.8 − 83.33 = 1240.4 $N/cm^2$

Total Stress at D = −1922.61 − 83.33 = −2005.94 $N/cm^2$

② A hook carries a load of 7.5kN and load lying is at a distance of 20mm from inner edge in the section. which is rectangular. The load also passes through the centre of curvature of hook. The diemension of rectangular section are 15mm width & 30mmdepth. Calculate Max & Min stress & plot variation of stress cross section

Solution

$r_i = 20mm, h = 30, b = 15$ (mm)

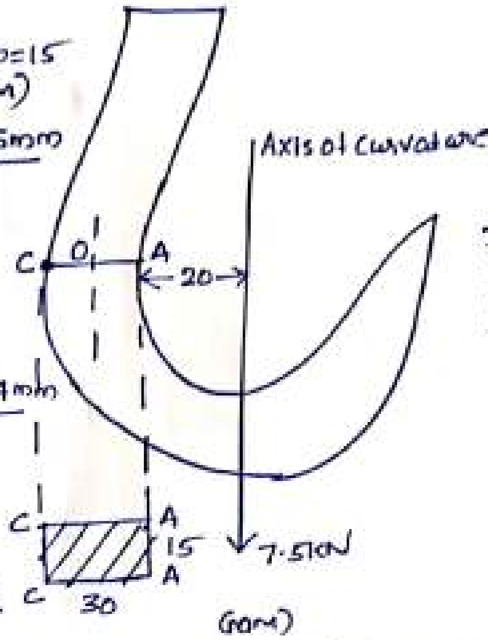
$r_c = r_i + \frac{h}{2} = 20 + \frac{30}{2} = 35mm$

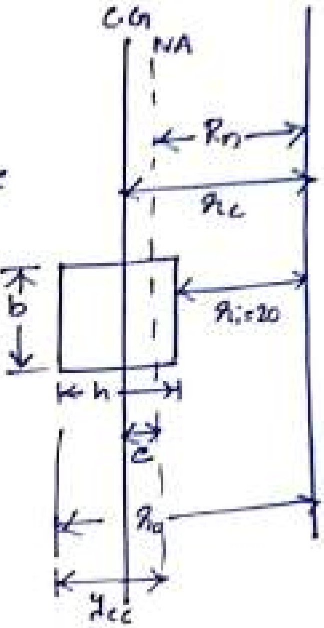
$R_n = \frac{h}{\ln\left(\frac{r_c + h/2}{r_c - h/2}\right)}$

$R_n = \frac{30}{\ln\left(\frac{35+30/2}{35-30/2}\right)} = 32.74mm$

$e = r_c - R_n$

$= 35 - 32.74 = 2.26cm$




Moment at O => $M = 7.5\times10^3(20+15) = 262.5\times10^3$ NmM

$A = 30\times15 = 450mm^2$

Section C-C

Tension

$\sigma_x = \frac{\pm M}{Ae}\frac{y}{R_n - y} = \frac{262.5\times10^3}{450\times2.26}\left[\frac{-17.26}{32.74-(-17.26)}\right]$

$= -84.100 \ N/mm^2$

$y_{cc} = -(e + h/2)$
$= -(15+2.26)$
$= -17.26mM$

Section A-A

$\sigma_x = \frac{M}{Ae}\frac{y}{R_n - y} = \frac{262.5\times10^3}{450\times2.26}\left[\frac{12.74}{32.74-12.74}\right]$

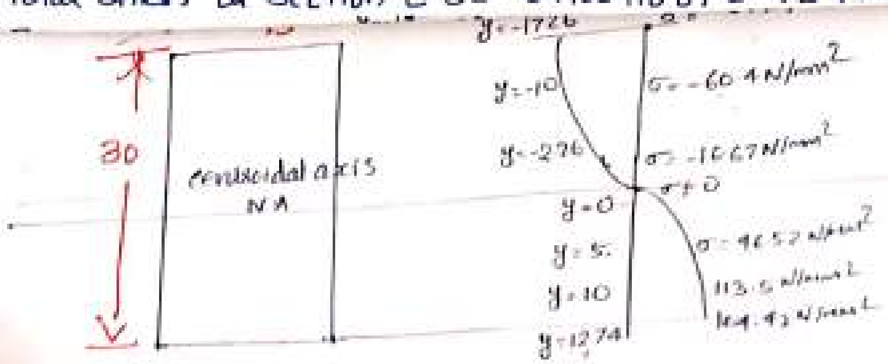
$= 164.417 \ N/mm^2$

$y_{AA} = 15 - 2.26$
$= 12.74mM$

Direct stress $= \frac{P}{A} = \frac{7.5\times10^3}{450} = 16.66 \ N/mm^2$

Total Stress at Section A-A = 164.417 + 16.67 = 181.09 $N/mm^2$

Total stress at Section C-C = -84.100 + 16.67 = -72.43 $N/mm^2$

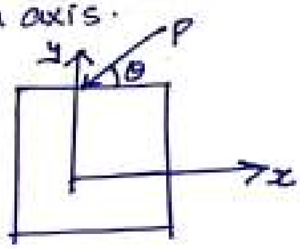


# Unsymmetrical Bending

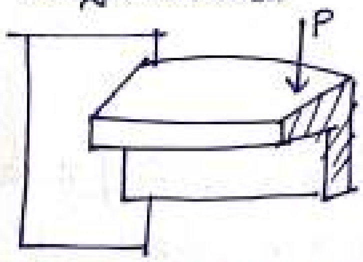
- In general bending / structural formula is $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$
- All the cross section have two axis passing through the centroid such that the moment of inertia about one of them is maximum and other is minimum. These are called principle axis of crosssection.
- If the load line on a straight beam does not coincide with one of the principal axis, the bending take place in a plane different from the principal plane. This type of bending is known as unsymmetrical bending.
- In case of unsymmetrical bending direction of neutral axis is not perpendicular to loading.

## Reason For unsymmetrical Bending

① The section is symmetrical but load inclined to both axis.



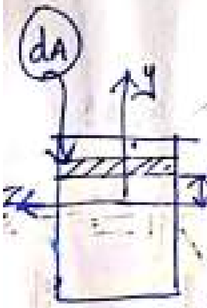

② The load is ⊥cular to the axis. But the section itself may be unsymmetrical.



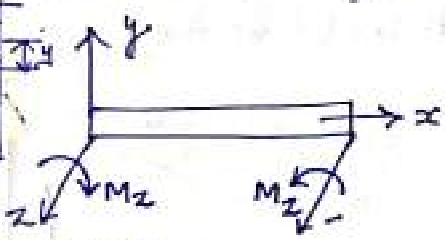

## STEPS TO FINDOUT BENDING STRESS OR UNSYMMETRICAL BENDING

The stress developed by bending formula is

$$\sigma = \frac{My}{I}$$

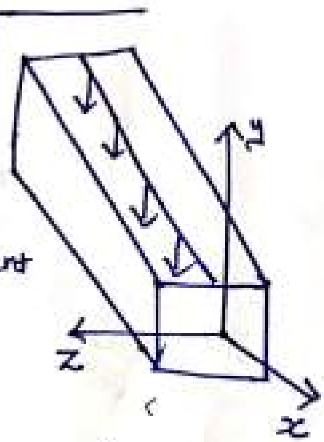


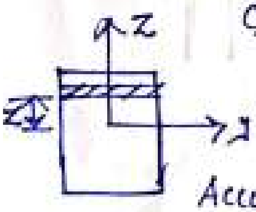

Case-I Applying moment in negative z-direction




The stress on x-axis by the application of above bending moment

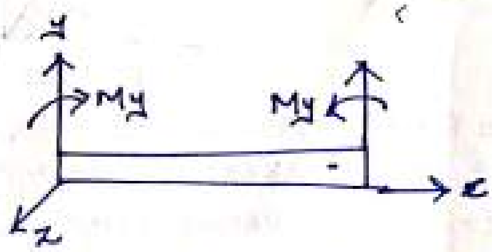
$$\sigma_x = \frac{-M_z y}{I_z}$$




Cas-II Applying BM in +y direction

$$\sigma_x = \frac{+M_y z}{I_y}$$



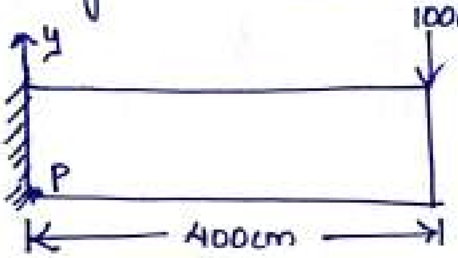

According to principle of superposition total stress on x axis will be equal to algebraic sum of above 2 stresses

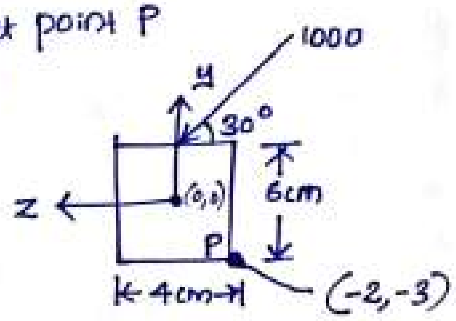
$$\boxed{\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}}$$

Orientation of Neutral axis w.r.t. to y

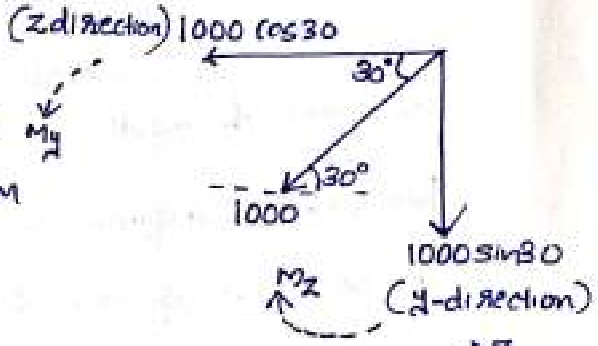

$\Rightarrow$ $\boxed{\tan\theta = \frac{M_z I_y}{M_y I_z}}$

① A cantilever beam of rectangular crosssection is subjected to a load of 1000 N at free end with an inclination of 30° as shown in fig. Find bending stress developed at point P




Solution



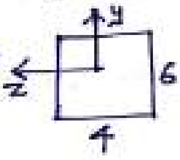

$M_y$ = Force in z direction × length of beam

$M_z$ = Force in y direction × Length of beam

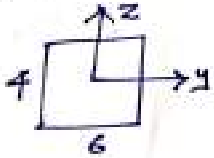
$M_y = -1000\cos 30 \times 400 = -346400\ \text{Ncm}$

$M_z = -1000\sin 30 \times 400 = -200000\ \text{Ncm}$

$I_z = \dfrac{4\times 6^3}{12} = 72\,\text{cm}^4$




$I_y = \dfrac{6\times 4^3}{12} = 32\,\text{cm}^4$



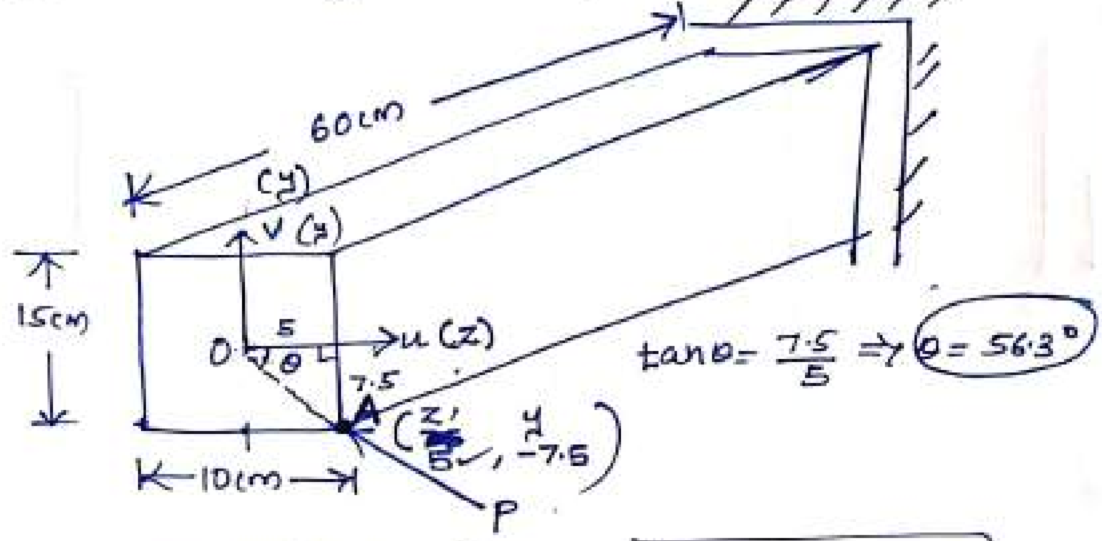

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

Here we have to find stress at $P(-2,-3)$ $[z=-2\,\text{cm}, y=-3\,\text{cm}]$

$$P(-2,-3) \Rightarrow \sigma_x = \frac{-346400\times -2}{32} - \frac{(-200000\times -3)}{72} = 13316.66\ \text{N/cm}^2$$

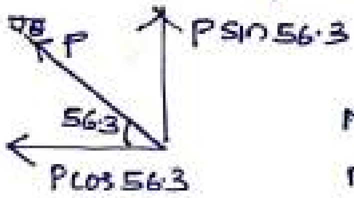
$$\tan\theta = \frac{M_z I_y}{M_y I_z} = \frac{-200000\times 32}{-346400\times 72} \Rightarrow \theta = 14.4^\circ$$

② Find the Value of Load P in fig①. So that Maximum bending stress allowded is 15Mpa, for the case of beam given below, Subjected to unsymmetrical bending.




$\tan\theta = \frac{7.5}{5} \Rightarrow \theta = 56.3°$

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z} \Rightarrow \boxed{\sigma_x = \frac{M_y z}{I_y} + \frac{M_z y}{I_z}}$$


$M_y$ = Force in z dire × length

$M_y = -P\cos 56.3 \times 0.6$

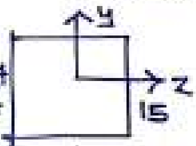
$M_y = -0.3329\,P$

$M_z$ = Force in y dire × length

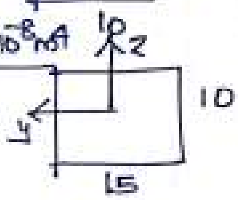
$= +P\sin 56.3 \times 0.6$

$M_z = 0.49917\,P$

$I_z = \frac{10 \times 15^3}{12} = 2812.5 \times 10^{-8}\ m^4$




$I_y = \frac{15 \times 10^3}{12} = 1250 \times 10^{-8}\ m^4$




$\sigma_x = 15\,Mpa = 15 \times 10^6\ N/m^2$

$$\sigma_x = \frac{M_y z}{I_y} + \frac{M_z y}{I_z}$$

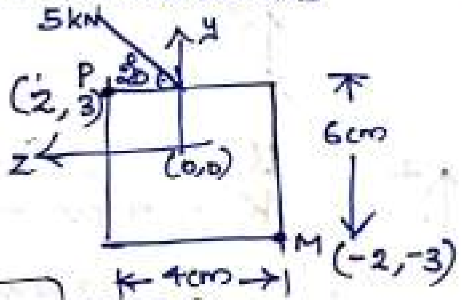
$$15 \times 10^6 = \frac{-0.3329\,P \times 5 \times 10^{-2}}{1250 \times 10^{-8}} + \frac{0.4991\,P \times -7.5 \times 10^{-2}}{2812.5 \times 10^{-8}}$$

$$15 \times 10^6 = -1331.64\,P - 1331.12\,P$$

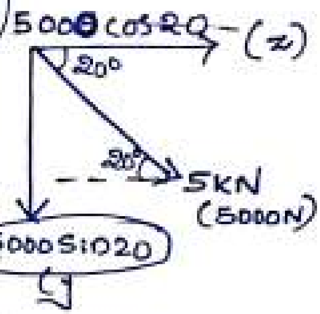
$P = -5633.25\ N$

$P = 5.63\ kN$

③ A cantilever of rectangular crosssection of length 2m, breadth 4cm is subjected to an inclined load of 5kN at free end with inclination of 20° with vertical axis as shown in fig. What is the maximum value of Bending stress and also find inclination with neutral axis


$$\boxed{\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}}$$



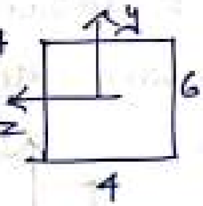

$M_y$ = Force in z direction x Length

$M_y = 5000\cos 20 \times 200 = 939692.6$ Ncm

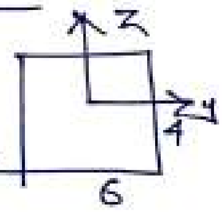
$M_z$ = Force in y direction x Length

$= -5000\sin 20 \times 200 = -342020.14$ Ncm

$I_z = \frac{4\times6^3}{12} = 72\ cm^4$




$I_y = \frac{6\times4^3}{12} = 32\ cm^4$




$\sigma_x$ at point P (2, 3) — (z, y)

$$\sigma_x = \frac{939692.6\times 2}{32} - \left(\frac{-342020.14\times 3}{72}\right)$$

$$= 72981.62\ N/cm^2$$

$\sigma_x$ at point M $(-2, -3)$ — (z, y)

$$\sigma_x = \frac{939692.6\times -2}{32} - \left(\frac{-342020.14\times -3}{72}\right)$$

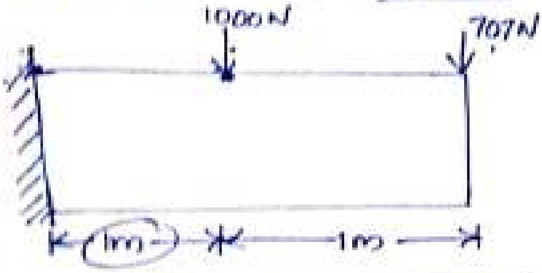
$$= -72981.62\ N/cm^2$$

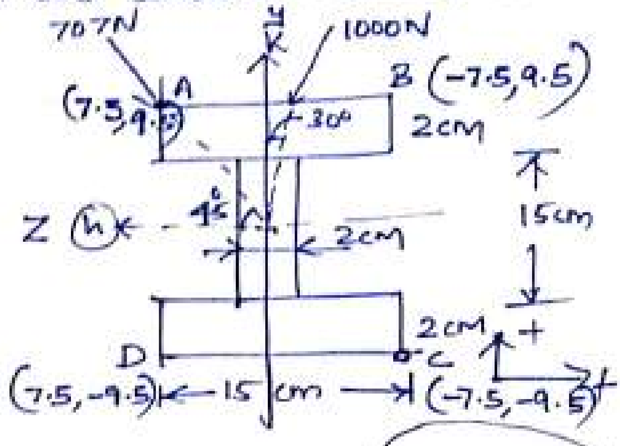
Max value $= 72981.62\ N/cm^2$

$$\tan\theta = \frac{M_z I_y}{M_y I_z} = \frac{-342020.6\times 32}{939692.6\times 72}$$

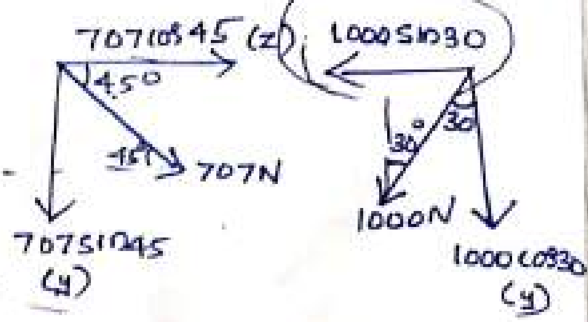
$$\theta = -9.18^\circ$$

(4) (1) A cantilever beam of I section is used to support the loads inclined to the axis as shown in fig. Compute the stress at corner ABCD of the section and what is inclination of neutral axis with z axis





$$\boxed{\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}}$$



$M_z$ = Force in y direction x length

$M_z = -707\sin 45 \times 2 - 1000\cos 30 \times 1$

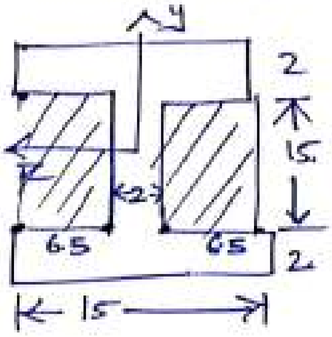
$M_z = \underline{-1865.847\ Nm}$

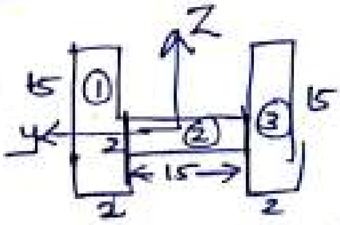
$M_y$ = Force in z direction x length

$M_y = 707\cos 45 \times 2 - 1000\sin 30 \times 1 \Rightarrow M_y = \underline{500\ NM}$

$I_z = \frac{15 \times 19^3}{12} - \frac{13 \times 15^3}{12}$

$I_z = \underline{4917.5\ cm^4}$





$I_y = \frac{2 \times 15^3}{12} + \frac{15 \times 2^3}{12} + \frac{2 \times 15^3}{12}$

$I_y = \underline{1135\ cm^4}$

at A $(\overset{z}{7.5}, \overset{y}{9.6})$

$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z} = \left(\frac{500 \times 10^2 \times 7.5}{1135} - \left(\frac{-1865.8 \times 10^2 \times 9.5}{4917.5}\right)\right)$

$\sigma_x = \underline{690.9\ N/cm^2}$

at B $(\overset{z}{-7.5}, \overset{y}{9.5})$

$\sigma_x = \frac{500 \times 10^2 \times -7.5}{1135} - \left(\frac{-1865.87 \times 10^2 \times 9.5}{4917.5}\right) = \underline{30.1\ N/cm^2}$

at C $(-7.5, -9.5) \Rightarrow \sigma_x = \frac{500 \times 10^2 \times -7.5}{1135} - \left(\frac{-1865.67 \times 10^2 \times -9.5}{4917.5}\right)$

$= \underline{-690.9\ N/cm^2}$

at D $(7.5, -9.5) \Rightarrow \sigma_x = \frac{500 \times 10^2 \times 7.5}{1135} - \left(\frac{-1865.87 \times 10^2 \times -4.5}{4917.5}\right) = \underline{-30.1\ N/cm^2}$

$\tan\beta = \frac{M_y I_z}{M_z I_y} = \frac{500 \times 10^2 \times 4917.5}{-1865.87 \times 10^2 \times 1135} \Rightarrow \underline{\beta = 49.26^\circ}$

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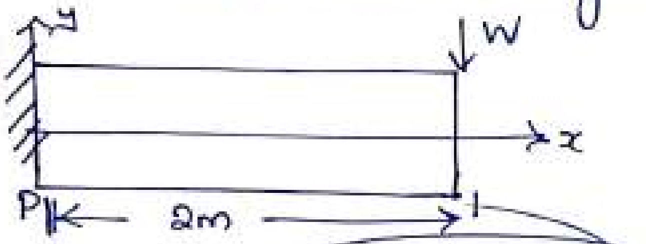
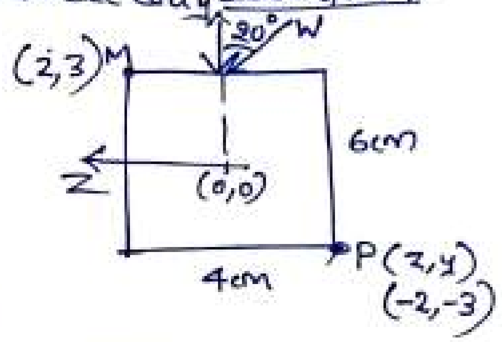
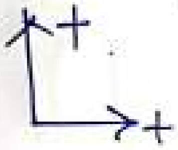
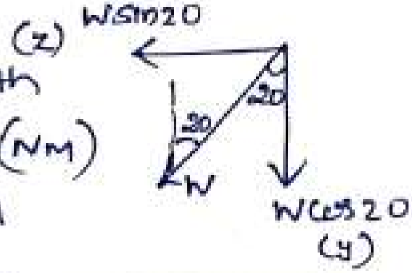
(5) A cantilever of rectangular crosssection of breadth 4cm and depth 6cm is subjected to inclined load W at free end. The length of cantilever is 2m and angle of inclination of load with vertical 20°. What is maximum value of W. If the max stress due to bending not to exceed 200 N/mm².




$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$




$M_y$ = Force in z direction × length

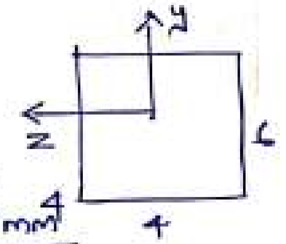
$M_y = -W\sin 20 \times 2 = -0.684\,W\,(Nm)$

$M_y = -0.684 \times 10^3\,W\ Nmm$

$M_z$ = Force in y direction × length

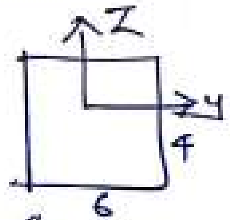
$= -W\cos 20 \times 2 = -1.879\,W\,(Nm) = -1.879 \times 10^3\,W\ Nmm$

$I_z = \frac{4 \times 6^3}{12} = 72\ cm^4$




$I_z = 72 \times 10^4\ mm^4$

$I_y = \frac{6 \times 4^3}{12} = 32\ cm^4$




$I_y = 32 \times 10^4\ mm^4$

at M $\left(\overset{z}{2}, \overset{y}{3}\right)$ cm $\Rightarrow$ at M $\left(\overset{z}{20}, \overset{y}{30}\right)$ mm

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z} = \frac{-0.684 \times 10^3 \times 20}{32 \times 10^4} - \left(\frac{-1.879 \times 10^3 \times 30}{72 \times 10^4}\right)$$

$\sigma_x = 200\ N/mm^2$

$\sigma_x = -0.043W + 0.078W \Rightarrow 200 = -0.043W + 0.078W$

$W = 5714\ N$

at P $\left(\overset{z}{-2}, \overset{y}{-3}\right)$ cm $\Rightarrow$ at P $\left(\overset{z}{-20}, -30\right)$ mm

$\sigma_x = 200\ N/mm^2$

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z} = \frac{-0.684 \times 10^3 \times -20}{32 \times 10^4} - \left(\frac{-1.87 \times 10^3 \times -30}{72 \times 10^4}\right)$$

$\sigma_x = 0.043W - 0.078W \Rightarrow 200 = \pm 0.035W$

$W = -5714\ N$

$$\tan\beta = \frac{M_y I_z}{M_z I_y} = \frac{-0.684 \times 10^3 \times 5714 \times 72}{-1.879 \times 10^3 \times 5714 \times 32} \Rightarrow \beta = -35.8^\circ$$

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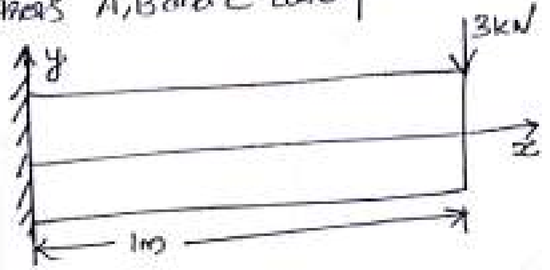
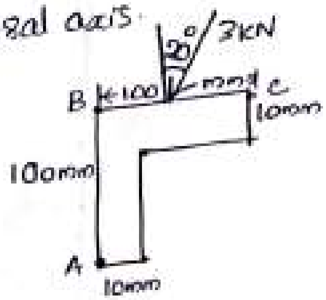
③ A Cantilever of angle section is 1m long and fixed at one end and is subjected to load of 3kN at free end 20° to the Vertical. calculate Bending Stress A, B and C. also position of neutral axis.

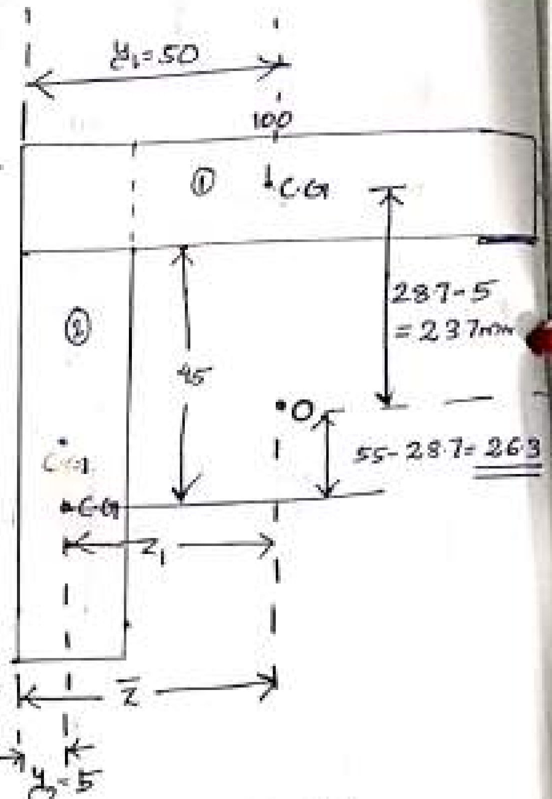



Solution

$$\bar{z}=\bar{y}=\frac{A_1y_1+A_2y_2}{A_1+A_2}=\frac{100\times10\times50+90\times10\times5}{100\times10+90\times10}$$

$= \underline{28.7mm}$




$$I_z=I_y=\left[\frac{100\times10^3}{12}+100\times10\times23.7^2\right]+\left[\frac{10\times90^3}{12}+10\times90\times26.3^2\right]$$

$= \underline{1.8\times10^6mm^4}$

$I_{zy}=A_1\bar{z}_1\bar{y}_1+A_2\bar{z}_2\bar{y}_2$

$=(100\times50)\times(50-28.5)\times23.7$

$+90\times10(-23.7)\times(-26.3)$

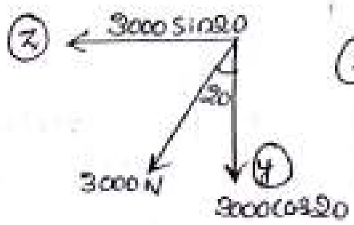
$= 1.074\times10^6mm^4$ //

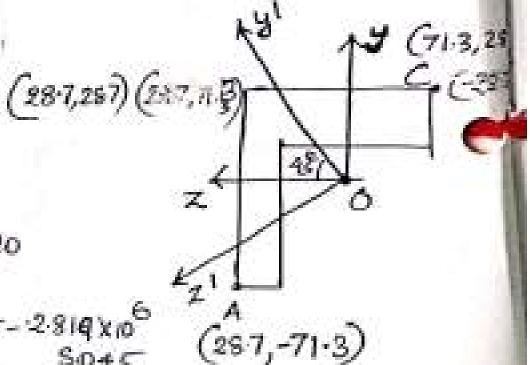
$M_z=-3000\cos20\times1000$

$= -2.819\times10^6Nmm$

$M_y=-3000\sin20\times1000$

$= -1.026\times10^6Nmm$




$M_{y'}=M_y\cos\alpha+M_z\sin\alpha=-1.026\times10^6\cos45-2.819\times10^6\sin45$

$= -2.719\times10^6Nmm$ //

$M_{z'}=M_z\cos\alpha-M_y\sin\alpha$

$= -1.268\times10^6Nmm$

$y'=y\cos45°+z\sin45°$

$= -71.3\cos45+28.7\sin45$

$= -30.12mm$

$z'=z\cos45-y\sin45$

$= 28.7\cos45-(-71.3\sin45)$

$= 70.498mm$

$$\sigma_x=\frac{M_{y'}z'}{I_{y'}}-\frac{M_{z'}y'}{I_{z'}}$$

where $I_{y'}=\frac{I_z+I_y}{2}-\sqrt{\left(\frac{I_z-I_y}{2}\right)^2+I_{zy}^2}$

$=\frac{1.8\times10^6\times2}{2}-\sqrt{0^2+(1.074\times10^6)^2}=\underline{0.726\times10^6mm^4}$

$I_{z'}=\frac{I_z+I_y}{2}+\sqrt{\left(\frac{I_z-I_y}{2}\right)^2+I_{zy}^2}=\frac{1.8\times10^6\times2}{2}+\sqrt{0^2+1.074\times10^6}=\underline{2.874\times10^6mm^4}$

$$\sigma_x = \frac{M_{y'} z'}{I_{y'}} - \frac{M_{z'} y'}{I_{z'}}$$

at $A(287, -71.3)$

$$\sigma_x = \frac{-2.719\times10^6 \times 70.498}{0.726\times10^6} - \frac{(-1.268\times10^6 \times -30.12)}{2.874\times10^6}$$

$$= \underline{\underline{-265.01\,\text{Mpa}}}\ (N/mm^2)$$

Similarly, $B(\overset{z}{287}, \overset{y}{287})$

$$\Rightarrow y' = y\cos45 + z\sin45$$
$$= 287\cos45 + 287\sin45 = \underline{\underline{40.58\,mm}}$$
$$\Rightarrow z' = z\cos45 - y\sin45$$
$$= 287\cos45 - 287\sin45 = \underline{\underline{0\,mm}}$$

$$\sigma_x = \frac{-2.719\times10^6 \times 0}{0.726\times10^6} - \frac{(-1.268\times10^6 \times 40.58)}{2.874\times10^6} = 17.9\,\text{Mpa}$$

Similarly $C(\overset{z}{71.3}, \overset{y}{287})$

$$\Rightarrow y' = y\cos45 + z\sin45$$
$$= 287\cos45 + 71.3\sin45 = 70.71\,mm$$
$$\Rightarrow z' = z\cos45 - y\sin45$$
$$= 71.3\cos45 - 287\sin45 = 30.12\,mm$$

$$\sigma_x = \frac{-2.719\times10^6 \times 30.12}{0.726\times10^6} - \frac{(-1.268\times10^6 \times 70.71)}{2.874\times10^6} = \underline{\underline{251.5\,\text{Mpa}}}$$

$$\tan\beta = \frac{-M_{y'} I_{z'}}{M_{z'} I_{y'}} = \frac{-2.719\times10^6 \times 2.874\times10^6}{1268\times10^6 \times 0.726\times10^6} = \underline{\underline{-8.4886}}$$

$$\beta = \underline{\underline{-83.3^\circ}}$$

Inclination of neutral axis with z-axis $= -83.3 + 45 = \underline{\underline{-38.3^\circ}}$