

Module -2

Constitutive Equations

The equations relating the components of stress and strain are called Constitutive equations:

STRESS - STRAIN EQUATION	
$\sigma_{xx} = a_{11}\epsilon_{xx} + a_{12}\epsilon_{yy} + a_{13}\epsilon_{zz} + a_{14}\tau_{xy} + a_{15}\tau_{yz} + a_{16}\tau_{zx}$ $\sigma_{yy} = a_{21}\epsilon_{xx} + a_{22}\epsilon_{yy} + a_{23}\epsilon_{zz} + a_{24}\tau_{xy} + a_{25}\tau_{yz} + a_{26}\tau_{zx}$ $\sigma_{zz} = a_{31}\epsilon_{xx} + a_{32}\epsilon_{yy} + a_{33}\epsilon_{zz} + a_{34}\tau_{xy} + a_{35}\tau_{yz} + a_{36}\tau_{zx}$ $\epsilon_{xy} = a_{41}\epsilon_{xx} + a_{42}\epsilon_{yy} + a_{43}\epsilon_{zz} + a_{44}\tau_{xy} + a_{45}\tau_{yz} + a_{46}\tau_{zx}$ $\epsilon_{yz} = a_{51}\epsilon_{xx} + a_{52}\epsilon_{yy} + a_{53}\epsilon_{zz} + a_{54}\tau_{xy} + a_{55}\tau_{yz} + a_{56}\tau_{zx}$ $\epsilon_{zx} = a_{61}\epsilon_{xx} + a_{62}\epsilon_{yy} + a_{63}\epsilon_{zz} + a_{64}\tau_{xy} + a_{65}\tau_{yz} + a_{66}\tau_{zx}$	$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$

Equation Form

Matrix Form

Strain - Stress Relation

EQUATION FORM	MATRIX FORM
$\epsilon_{xx} = b_{11}\sigma_x + b_{12}\sigma_y + b_{13}\sigma_z + b_{14}\tau_{xy} + b_{15}\tau_{yz} + b_{16}\tau_{zx}$ $\epsilon_{yy} = b_{21}\sigma_x + b_{22}\sigma_y + b_{23}\sigma_z + b_{24}\tau_{xy} + b_{25}\tau_{yz} + b_{26}\tau_{zx}$ $\epsilon_{zz} = b_{31}\sigma_x + b_{32}\sigma_y + b_{33}\sigma_z + b_{34}\tau_{xy} + b_{35}\tau_{yz} + b_{36}\tau_{zx}$ $\tau_{xy} = b_{41}\sigma_x + b_{42}\sigma_y + b_{43}\sigma_z + b_{44}\tau_{xy} + b_{45}\tau_{yz} + b_{46}\tau_{zx}$ $\tau_{yz} = b_{51}\sigma_x + b_{52}\sigma_y + b_{53}\sigma_z + b_{54}\tau_{xy} + b_{55}\tau_{yz} + b_{56}\tau_{zx}$ $\tau_{zx} = b_{61}\sigma_x + b_{62}\sigma_y + b_{63}\sigma_z + b_{64}\tau_{xy} + b_{65}\tau_{yz} + b_{66}\tau_{zx}$	$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$

Equation Form

Matrix Form

Above two relations linear elastic materials the coefficient $a_{11}, a_{12}, \dots, b_{11}, b_{12}, \dots$ are constants. Above set of eq: known as Generalized Hook's Law for Homogeneous linearly Elastic Materials

- ① What is constitutive eq:

C Generalise Hooke's Law for Isotropic Materials

Stress - Strain Relation of Isotropic Materials

Isotropic Material: Material property of solid has constant value along all directions.

Material linear elastic isotropic with respect to Elastic behaviour

Stress-strain eq: written using two properties.

Commonly used Material property

- Young Modulus [E]
- Modulus of Rigidity [G]
- Poisson Ratio (ν)
- Bulk Modulus (K)
- Lamé Coefficient μ and λ

Assignment
Derive Lamé's Coefficient (μ, λ)
x Bulk Modulus (Derivation)

Stress - Strain Relation

$$\sigma_{xx} = (2\mu + \lambda) \epsilon_{xx} + \lambda \epsilon_{yy} + \lambda \epsilon_{zz}$$

$$\sigma_{yy} = (2\mu + \lambda) \epsilon_{yy} + \lambda \epsilon_{xx} + \lambda \epsilon_{zz}$$

$$\sigma_{zz} = (2\mu + \lambda) \epsilon_{zz} + \lambda \epsilon_{xx} + \lambda \epsilon_{yy}$$

$$\tau_{xy} = \mu \gamma_{xy}, \quad \tau_{yz} = \mu \gamma_{yz}, \quad \tau_{zx} = \mu \gamma_{zx}$$

Eq: Form

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}$$

$$\nu = \frac{\lambda}{2(\mu + \lambda)}$$

$$\lambda = \frac{2\nu}{1-2\nu} \times \frac{E}{2(1+\nu)}$$

$$G = \mu = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-2\nu)}$$

$$\frac{3}{G} + \frac{1}{K} = \frac{9}{E}$$

Relation of E, G, K
(Derivation)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Matrix Form

$$\text{or} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\frac{2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Strain - Stress Relation

$$\epsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz}))$$

$$\epsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz}))$$

$$\epsilon_{zz} = \frac{1}{E} (\sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy}))$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

eq 1 & 2 Matrix Form
Above set of Eq: known as Generalise
Hooke's law of Isotropic material

For given tensorial strain at a point on steel object $E = 207 \times 10^6 \text{ kPa}$, $G_1 = 80 \times 10^6 \text{ kPa}$

Determine the Stress matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & \frac{3}{2} \\ -1 & \frac{3}{2} & 0 \end{bmatrix} \times 10^3$$

Solution

Given $E = 207 \times 10^6 \text{ kPa}$, $G_1 = 80 \times 10^6 \text{ kPa}$

$$(\epsilon) = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & \frac{3}{2} \\ -1 & \frac{3}{2} & 0 \end{bmatrix} \times 10^3 \quad , \quad G_1 = H = 80 \times 10^6$$

- $\sigma_{xx} = 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$
 $= 2 \times 80 \times 10^6 \times 1 \times 10^{-3} + 114.15 \times 10^6 (1 - 3 + 0) \times 10^{-3}$
 $= \underline{-68.3 \times 10^3 \text{ kPa}}$

- $\sigma_{yy} = 2\mu\epsilon_{yy} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$
 $= 2 \times 80 \times 10^6 \times -3 \times 10^{-3} + 114.15 \times 10^6 (1 - 3 + 0) \times 10^{-3}$
 $= \underline{-708.3 \times 10^3 \text{ kPa}}$

- $\sigma_{zz} = 2\mu\epsilon_{zz} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$
 $= 2 \times 80 \times 10^6 \times 0 \times 10^{-3} + 114.15 \times 10^6 (1 - 3 + 0) \times 10^{-3} = \underline{-228.3 \times 10^3 \text{ kPa}}$

- $\tau_{xy} = \mu\gamma_{xy} = 80 \times 10^6 \times 0 = \underline{0}$, $\tau_{yz} = \mu\gamma_{yz} = 80 \times 10^6 \times \frac{3}{2} \times 10^{-3} = \underline{120 \times 10^3 \text{ kPa}}$
 $\tau_{zx} = \mu\gamma_{zx} = 80 \times 10^6 \times -1 \times 10^{-3} = \underline{-80 \times 10^3}$

Stress Matrix

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -68.3 & 0 & -80 \\ 0 & -708.3 & 120 \\ -80 & 120 & -228.3 \end{bmatrix} \times 10^3 \text{ kPa}$$

- (2) The state of stress at a point is given by matrix
 Compute strain tensor for matrix with
 $E = 207 \times 10^6 \text{ kPa}$ and $G_1 = 80 \times 10^6 \text{ kPa}$

Solution

Given

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 31 \end{bmatrix} \times 10^3 \text{ kPa}$$

$$\epsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) = \frac{1}{207 \times 10^6} (18 \times 10^3 - 0.2937(-50 + 31)) \times 10^3 = \underline{-0.1139 \times 10^3}$$

$$\epsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) = \frac{1}{207 \times 10^6} [-50 - 0.2937(18 + 31)] \times 10^3 = \underline{-0.311 \times 10^3}$$

$$\epsilon_{zz} = \frac{1}{E} (\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) = \frac{1}{207 \times 10^6} [-31 - 0.2937(18 - 50)] \times 10^3 = \underline{0.195 \times 10^3}$$

$$\gamma_{xy} = \gamma_{yz} = \frac{\tau_{xy}}{G_1} = 0, \gamma_{xz} = \frac{\tau_{xz}}{G_1} = \frac{24 \times 10^3}{80 \times 10^6} = 0.3 \times 10^{-3}, \gamma_{yz} = \gamma_{zy} = \frac{\tau_{yz}}{G_1} = 0$$

$$G_1 = \frac{E}{2(1+\nu)} = \frac{80 \times 10^6}{2(1+\nu)} = 207 \times 10^6$$

$$2+2\nu = 207/80 \Rightarrow \nu = 0.294$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{0.294 \times 207 \times 10^6}{(1+0.294)(1-2 \times 0.294)} = \underline{114.15 \times 10^6 \text{ kPa}}$$

$$\text{Strain Matrix} \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} -0.1139 & 0 & 0.3 \times 10^{-6} \\ 0 & -0.311 & 0 \\ 0.3 \times 10^{-6} & 0 & 0.14516 \end{bmatrix}$$

- ③ In a tension test of 25mm dia rod extension of 0.25mm occurs over gauge length 300mm and correspondingly dia decreased by 0.00595mm when load 100kN applied. Determine modulus of elasticity and poisson's ratio.

Solution

$$\text{Dia of rod, } D = 25\text{mm} \quad \therefore A = \frac{\pi}{4} D^2 = 490.625\text{mm}^2$$

$$L = 300\text{mm}, P = 100\text{kN}$$

$$\Delta L = 0.25\text{mm}$$

$$\Delta D = 0.00595\text{mm}$$

- Stress, $\sigma = \frac{P}{A} = \frac{100 \times 10^3}{490.625} = 203.82 \text{N/mm}^2$
- Strain $\epsilon_p = \frac{\Delta L}{L} = \frac{0.25}{300} = 8.33 \times 10^{-4}$
- $\nu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\Delta D/D}{\Delta L/L} = \frac{2.38 \times 10^{-4}}{8.33 \times 10^{-4}} = 0.286$
- Bulk Modulus $k = \frac{E}{3(1-2\nu)} = \frac{2.45 \times 10^5}{3(1-2 \times 0.286)} = 1.91 \times 10^5 \text{N/mm}^2$
- Modulus of Rigidity $G_1 = \frac{E}{2(1+\nu)} = \frac{2.45 \times 10^5}{2(1+0.286)} = 0.95 \times 10^5 \text{N/mm}^2$

- ④ When Cu wire 40mm dia subjected to axial load of 80kN. It reduces the dia by 0.00775mm. Modulus of Rigidity for wire $0.4 \times 10^5 \text{N/mm}^2$. Calculate poisson's ratio and modulus of elasticity of material

Solution

$$D = 40\text{mm}, A = \frac{\pi}{4} D^2 = \frac{\pi \times 40^2}{4} = 1256\text{mm}^2$$

$$P = 80\text{kN}, \Delta D = 0.00775\text{mm}, G_1 = 0.4 \times 10^5 \text{N/mm}^2$$

$$\text{Stress} = \frac{P}{A} = \frac{80 \times 10^3}{1256} = 63.69 \text{N/mm}^2, \text{ Lateral strain} = \frac{\Delta D}{D} = \frac{0.00775}{40} = 1.9375 \times 10^{-4}$$

$$\text{Longitudinal strain} = \frac{\sigma}{E} = \frac{63.69}{E}$$

$$\text{Lateral strain} = \nu \times \text{Longitudinal strain}$$

$$1.9375 \times 10^{-4} = \nu \times \frac{63.69}{E} \Rightarrow E = 5224.98 \nu$$

$$E = 2G_1(1+\nu) \Rightarrow 5224.98 \nu = 2 \times 0.4 \times 10^5 (1+\nu) \Rightarrow \nu = 0.069$$

Boundary Conditions

SET of Boundary condition for elasticity problem for externally applied traction case

$$\sigma_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z = F_x$$

$$\tau_{yx}n_x + \sigma_{yy}n_y + \tau_{yz}n_z = F_y$$

$$\tau_{zx}n_x + \tau_{zy}n_y + \sigma_{zz}n_z = F_z$$

(Applicable only surface load)

where F_x, F_y & F_z component of externally applied traction at boundary point

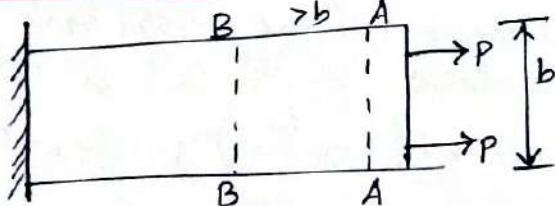
P. (Internal resisting traction Match with external applied traction as boundary condition)

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

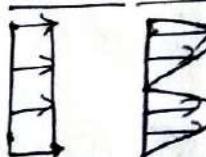
Saint Venant's principle

Statement The principle states that a change of loading distribution by a static equivalent system of forces having same resultant force and couple on a small part of the surface of the body would give rise to localized change in stress and strain only. Sufficiently away from this area the stress and strain field will not be affected.

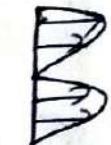
Example①



Stress distributions



Section B-B



Section A-A

Example②

* Consider a cantilever beam loaded with Force F

* It lead bending moment at A distance x from Force

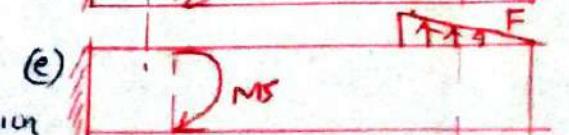
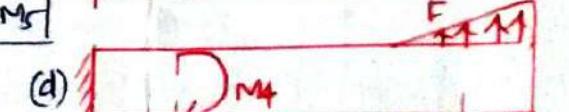
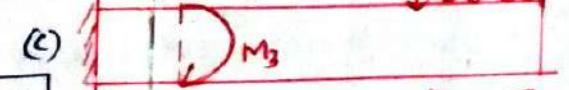
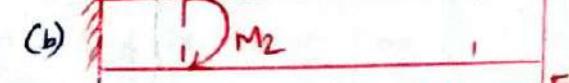
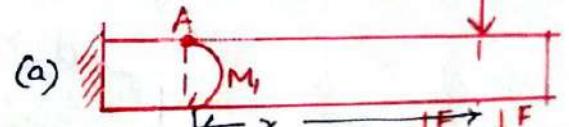
* Consider Fig(b) to (e)

According to Saint Venant's principle

$$\text{moment at A} \Rightarrow M_1 = M_2 = M_3 = M_4 = M_5$$

* Some local disturbance created by the end load on the stress near to the region of load application

* Variation die out quickly out side region



Uniqueness Theorem

Statement

Any physically realistic elasticity problem, defined by a set of governing equation and boundary conditions will have one and only one solution.

Proof : Assume two stress field satisfying equilibrium eq & Boundary condition

1st set
solution
field

$$\left. \begin{aligned} \frac{\partial}{\partial z} \sigma_{xx} + \frac{\partial}{\partial y} \tau_{xy} + \frac{\partial}{\partial z} \tau_{xz} + b_x = 0 \\ \frac{\partial}{\partial x} \tau_{yx} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \tau_{yz} + b_y = 0 \\ \frac{\partial}{\partial x} \tau_{zx} + \frac{\partial}{\partial y} \tau_{zy} + \frac{\partial}{\partial z} \sigma_{zz} + b_z = 0 \end{aligned} \right\} \quad (1)$$

Equilibrium eq

$$\left. \begin{aligned} \sigma_{xx} n_x + \tau_{xy} n_y + \tau_{xz} n_z = F_x \\ \tau_{yx} n_x + \sigma_{yy} n_y + \tau_{yz} n_z = F_y \\ \tau_{zx} n_x + \tau_{zy} n_y + \sigma_{zz} n_z = F_z \end{aligned} \right\} \quad (2)$$

Boundary condition

2nd set solution Field

$$\left. \begin{aligned} \frac{\partial}{\partial x} \sigma_{xx}^1 + \frac{\partial}{\partial y} \tau_{xy}^1 + \frac{\partial}{\partial z} \tau_{xz}^1 + b_x = 0 \\ \frac{\partial}{\partial x} \tau_{xy}^1 + \frac{\partial}{\partial y} \sigma_{yy}^1 + \frac{\partial}{\partial z} \tau_{yz}^1 + b_y = 0 \\ \frac{\partial}{\partial x} \tau_{xz}^1 + \frac{\partial}{\partial y} \tau_{zy}^1 + \frac{\partial}{\partial z} \sigma_{zz}^1 + b_z = 0 \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \sigma_{xx}^1 n_x + \tau_{xy}^1 n_y + \tau_{xz}^1 n_z = F_x \\ \tau_{yx}^1 n_x + \sigma_{yy}^1 n_y + \tau_{yz}^1 n_z = F_y \\ \tau_{zx}^1 n_x + \tau_{zy}^1 n_y + \sigma_{zz}^1 n_z = F_z \end{aligned} \right\} \quad (4)$$

$$\begin{aligned} \text{eq: } (1) - (3) \Rightarrow \frac{\partial}{\partial x} (\sigma_{xx} - \sigma_{xx}^1) + \frac{\partial}{\partial y} (\tau_{xy} - \tau_{xy}^1) + \frac{\partial}{\partial z} (\tau_{xz} - \tau_{xz}^1) = 0 \\ \frac{\partial}{\partial x} (\tau_{yx} - \tau_{yx}^1) + \frac{\partial}{\partial y} (\sigma_{yy} - \sigma_{yy}^1) + \frac{\partial}{\partial z} (\tau_{yz} - \tau_{yz}^1) = 0 \\ \frac{\partial}{\partial x} (\tau_{zx} - \tau_{zx}^1) + \frac{\partial}{\partial y} (\tau_{zy} - \tau_{zy}^1) + \frac{\partial}{\partial z} (\sigma_{zz} - \sigma_{zz}^1) = 0 \end{aligned} \quad (5)$$

Above eq: Shows difference b/w two stress field satisfy equilibrium with zero body force

$$\begin{aligned} \text{eq: } (2) - (4) \Rightarrow (\sigma_{xx} - \sigma_{xx}^1) n_x + (\tau_{xy} - \tau_{xy}^1) n_y + (\tau_{xz} - \tau_{xz}^1) n_z = 0 \\ (\tau_{yx} - \tau_{yx}^1) n_x + (\sigma_{yy} - \sigma_{yy}^1) n_y + (\tau_{yz} - \tau_{yz}^1) n_z = 0 \\ (\tau_{zx} - \tau_{zx}^1) n_x + (\tau_{zy} - \tau_{zy}^1) n_y + (\sigma_{zz} - \sigma_{zz}^1) n_z = 0 \end{aligned}$$

Difference of two stress field satisfy condition of zero external traction

- Strain energy & magnitude of external load
- Value of strain energy with difference of two stress field = zero
- Strain energy & square of stress at every point of solid
- Pointwise difference of two stress field = zero

Assumed two stress field satisfying set of eq case not covered

∴ There is one and only one stress field as solution

2-D Problem in Elasticity

A problem is in 2-D problem in Elasticity

If satisfy

- ① Body is prismatic [plane body have uniform thickness
Bounded by 2 plane || to xy plane]
- ② Body / Applied Force do not varies across thickness
- ③ Applied load cannot exist top & bottom of || surface
- ④ Surface force only lateral surface (no z-component)

2D problem in Elasticity

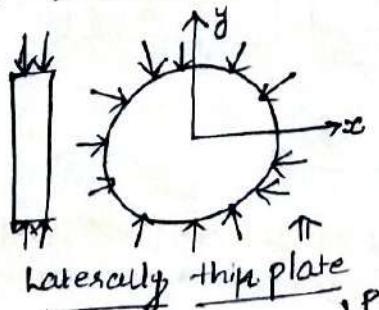
P

Plane Stress problem

Plane strain problem

- If thickness small compared to other dimension of parallel face
It is called plane stress problem
- A co-ordinate system x-z-z describing solid have $\sigma_{xx} = \sigma_{yy} = \tau_{xy} = 0$
 $\sigma_{xz}, \sigma_{yz}, \tau_{xy}$ are non zero

Example (a)



A thin cantilever \Rightarrow



- Above two fig: plane stress due to applied load uniformly throughout height / thickness || x-y plane
- Both Fig (a) & (b) cross section || x-y plane and same stress distribution

- If thickness large compared to other dimension of the parallel end faces
It is called plane strain problem

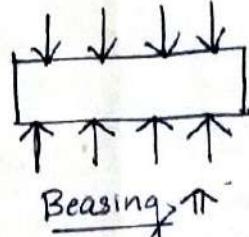
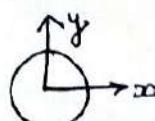
- There exist a co-ordinate system x-y-z for describing a solid

$$\epsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$

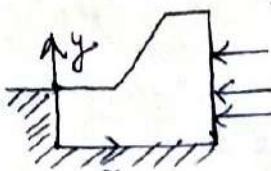
$$\epsilon_{xx} = \epsilon_{yy} = \gamma_{xy} = \text{non zero}$$

Example

(a)



(b)



- plane strain condition possible when displacement of particular direction arrested
- cross section || plane has same stress pattern.

Stress compatibility Equation of plane strain problem

VIMP
For derivation
Look Moon Note

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1-\nu) \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right]$$

where $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2$ called Laplacian operator

Stress compatibility Equation of plane strain problem

VIMP
For derivation
Look Moon Note

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\frac{1}{(1-\nu)} \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right]$$

Airy's Stress Function

Definition

Airy's Stress function is a scalar function ϕ with its derivative along the co-ordinate direction represent the component of stress that the equilibrium eq. are exactly satisfied.

where $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} - \nu$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} - \nu$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

✓ Related to body force

$$bx = \frac{\partial v}{\partial x} \quad by = \frac{\partial v}{\partial y}$$

Stress compatibility Equation in terms of Airy's Stress function ϕ

VIMP
For derivation
Look Moon Note

Plane Stress Case

$$\nabla^4 \phi = (1-\nu) \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right]$$

where $\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}$

For derivation
Look Moon Note

Plane Strain case

$$\Rightarrow \underbrace{\frac{\partial^4 \phi}{\partial x^2} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}}_{\nabla^4 \phi} = \left(\frac{1-2\nu}{1-\nu} \right) \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right)$$

Stress compatibility Equation are called Bi-harmonic equation

① Prove $\nabla^4 \phi = 0$ if (Body Force = 0)

Solution Compatibility eq. $\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0 \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) \phi = 0 \Rightarrow \nabla^2 \cdot \nabla^2 \phi = 0 \Rightarrow \nabla^4 \phi = 0$$

• Prove that the function $\phi = A(x^4 - 3x^2y^2)$ is possible stress function for a 2D elasticity problem. Find stress distribution due to this function.

Solution

$$\phi = A(x^4 - 3x^2y^2)$$

If it is possible, it satisfies bi-harmonic eq:

$$\text{where } \frac{\partial \phi}{\partial x} = A(4x^3 - 6x^2y) \quad \frac{\partial \phi}{\partial y} = A(-6x^2y)$$

$$\frac{\partial^2 \phi}{\partial x^2} = A(12x^2 - 6y^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} = -6Ax^2$$

$$\left[\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \right] \quad (1)$$

$$\frac{\partial^3 \phi}{\partial x^3} = A(24x)$$

$$\frac{\partial^3 \phi}{\partial y^3} = 0$$

$$\frac{\partial^4 \phi}{\partial x^4} = 24A$$

$$\frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial y^2} \right) &= \frac{\partial^2}{\partial x^2} (-6Ax^2) \\ &= \frac{\partial}{\partial x} (-12Ax) = -12A \end{aligned}$$

$$\text{Sub: Value in Bi-harmonic eq (1)} \Rightarrow 24A + 2(-12A) + 0 = 0 //$$

ϕ - Satisfy Bi-harmonic eq:

Stress component $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = -6Ax^2, \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = A(12x^2 - 6y^2)$

$$\tau_{xy} = -\frac{\partial \phi}{\partial x} = -\frac{\partial \phi}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = -\frac{\partial}{\partial x} (A(-6x^2y)) = 12Ax^2y$$

Polynomial Method of Solution

Polynomial of Degree = 1

By heart

$$\phi = a_1x + b_1y$$

Degree = 2

$$\phi = Ax^2 + Bxy + Cy^2$$

Degree = 3

$$\phi = Ax^3 + Bx^2y + Cxy^2 + Dx^3y$$

Degree = 4

$$\phi = Ax^4 + Bx^3y + Cx^2y^2 + Dx^4y^3 + E$$

• Polynomial of Degree = 1

$$\phi = a_1x + b_1y$$

$$\text{Bi-harmonic eq} \Rightarrow \left[\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \right]$$

Sub: Values we get Bi-harmonic eq = 0

Where

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 0, \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

• Polynomial of Degree = 2

$$\phi = Ax^2 + Bxy + Cy^2 \quad \text{where } A, B, C \text{ are constant}$$

$$\left[\frac{\partial \phi}{\partial x} = 2Ax + By \right]$$

$$\frac{\partial \phi}{\partial y} = Bx + 2Cy$$

$$\nabla^4 \phi = 0$$

$$\left[\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \right] \quad (1)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2A$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2C$$

$$\frac{\partial^3 \phi}{\partial x^3} = \frac{\partial^4 \phi}{\partial x^4} = 0$$

$$\frac{\partial^3 \phi}{\partial y^3} = 0$$

$$\frac{\partial^2}{\partial x} \left(\frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial^2}{\partial x} (2A) = 0$$

$$\frac{\partial^4 \phi}{\partial y^4} = 0$$

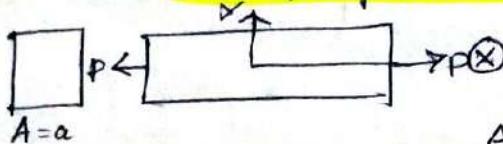
$$\text{Sub: eq} \Rightarrow \nabla^4 \phi = 0$$

Stress component

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2c, \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 2A, \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = -\frac{\partial}{\partial x} (Bx + 2A) = -B$$

Example : Straight beam Subjected axial

tension/compression in x direction only



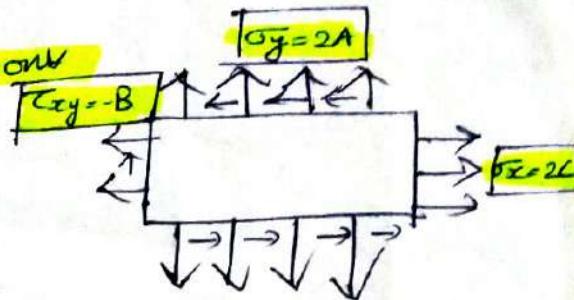
where

$$\sigma_y = \tau_{xy} = 0$$

$$A = B = 0$$

$$\sigma_x = 2c \Rightarrow \frac{P}{a} = 2c$$

$$\phi = \frac{P}{2a} y^2$$



• Polynomial of Degree three

$$\phi = Ax^3 + Bx^2y + Cxy^2 + Dy^3 \quad \text{where } A, B, C, D \text{ are constant}$$

B. has harmonic eq:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Sub: Values $\nabla^4 \phi = 0$

Stress components

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2Cx + 6Dy$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 6Ax + 2By, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = -\frac{\partial}{\partial x} (Bx^2 + 2Cxy + 3Dy^2) \\ = -(2Bx + 2Cy)$$

Example : pure Bending

• σ_x exist varies linearly in y direction

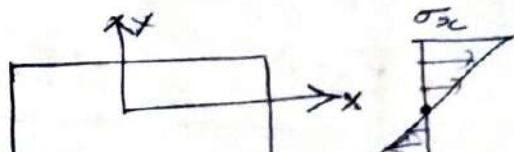
where

$$\phi = \frac{M}{6I} y^3$$

Here $A = B = C = 0$

$$\sigma_x = 6Dy$$

$$\tau_{xy} = \sigma_y = 0$$



$$\frac{M}{I} = \frac{\sigma_x}{y} \Rightarrow \sigma_x = \frac{My}{I} = 6Dy$$

• If σ_y exist varies y direction

$$\phi = \frac{M}{GI} x^3$$

Polynomial of Degree Four

$$\phi = Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4$$

Bi-harmonic eq:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Sub Value in Above eq:

$$24A + (2x^4c) + 24E = 0$$

$$24E = -24A - 8c$$

$$E = -(A + \frac{c}{3})$$

Stress components

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2Cx^2 + 6Dxy + 12Ey^2$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 12Ax^2 + 6Bxy + 2Cy^2$$

$$\tau_{xy} = -\frac{\partial^3 \phi}{\partial x \partial y} = -\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = -\frac{\partial}{\partial x} (Bx^3 + 12Cx^2y + 3Dxy^2 + 4Ey^3) = -\underline{(3Bx^2 + 4Cxy + 3Dy^2)}$$

Example (Loading) - Rectangular plate

(a) Except B

$$A=C=D=E=0$$

$$\begin{cases} \sigma_{xx}=0 \\ \sigma_{yy}=6Bxy \\ \tau_{xy}=-3Bx^2 \end{cases}$$

$$\frac{\partial \phi}{\partial x} = 4Ax^3 + 3Bx^2y + 2Cxy^2 + Dy^3$$

$$\frac{\partial \phi}{\partial x^2} = 12Ax^2 + 6Bxy + 2Cy^2$$

$$\frac{\partial^2 \phi}{\partial x^3} = 24Ax + 6By$$

$$\frac{\partial^4 \phi}{\partial x^4} = 24A$$

$$\frac{\partial \phi}{\partial y} = Bx^3 + 2Cx^2y + 3Dxy^2 + 4Ey^3$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2Cx^2 + 6Dxy + 12Ey^2$$

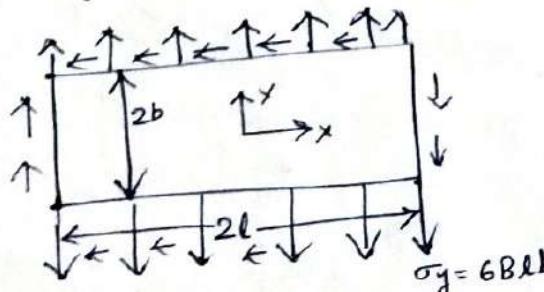
$$\frac{\partial^3 \phi}{\partial y^3} = 6Dx + 24Ey, \frac{\partial^4 \phi}{\partial y^4} = 24E$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial \phi}{\partial y} \right) = \frac{\partial^2}{\partial x^2} (2Cx^2 + 6Dxy + 12Ey^2) \\ = \frac{\partial}{\partial x} (4Cx + 6Dy + 0) = 4C$$

$$\sigma_{yy} = -\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial^2}{\partial x^2} (Bx^3 + 12Cx^2y + 3Dxy^2 + 4Ey^3) = -\underline{(3Bx^2 + 4Cxy + 3Dy^2)}$$

* Sides $x = \pm l$ (Shear stress distribute parabolically)

* other side shear stress uniformly distributed



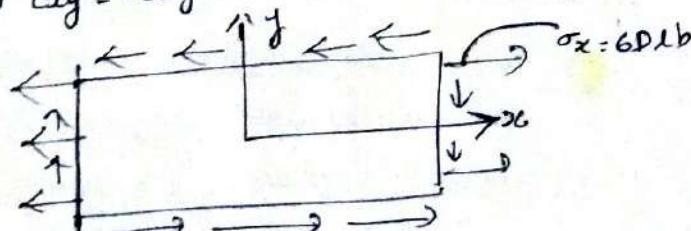
(b) Except D

$$A=B=C=E=0$$

$$\begin{cases} \sigma_{xx}=6Dxy \\ \sigma_{yy}=0 \\ \tau_{xy}=-3Dy^2 \end{cases}$$

* Sides $y = \pm b$ Shear stress distribute uniformly

* other side parabolically



• Polynomial of Degree - 5

$$\phi(x,y) = a_5x^5 + b_5x^4y + c_5x^3y^2 + d_5x^2y^3 + e_5xy^4 + f_5y^6$$

Bending of Cantilever Beam Loaded at End

- Consider cantilever beam - Narrow rectangular C.S.
- Force P applied at Free end

Assume $\phi = Axy^3 + Bxy$



upper & lower edge of beam
are free from load

Biharmonic eq: $\nabla^4 \phi = 0$

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Sub: Values $\Rightarrow \nabla^4 \phi = 0$

Stress component

$$\sigma_{xz} = \frac{\partial^2 \phi}{\partial y^2} = 6Axy$$

$$\tau_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right)$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$= -\frac{\partial}{\partial x} (3Axy^2 + Bx) = -\underline{(3Ay^2 + B)}$$

$$\frac{\partial \phi}{\partial x} = Ay^3 + By \quad \frac{\partial \phi}{\partial y} = 3Ay^2 + Bx$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^3 \phi}{\partial x^3} = \frac{\partial^4 \phi}{\partial x^4} = 0 \quad \frac{\partial^2 \phi}{\partial y^2} = 6Ax^2 + 0$$

$$\frac{\partial^3 \phi}{\partial y^3} = 6Ax$$

$$\frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial}{\partial x^2} (6Ax^2) = 0$$

Evaluation of Constants A & B

① $\tau_{xy}=0$ when $y=\pm b \Rightarrow 0 = -(3Ay^2 + B)$

$$B = -3Ab^2 \quad (y=b)$$

② Resultant of distributed Shear stress at any cross section P

$$\int_{-b}^b \tau_{xy} dy = P \Rightarrow \int_{-b}^b -(3Ay^2 + B) dy = P$$

Solving, we get

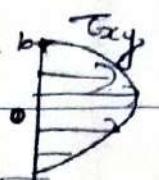
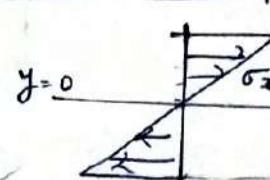
$$A = \frac{P}{4b^3}$$

$$B = -3Ab^2 = -\frac{3P}{4b}$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6Axy = \frac{Pxy}{\frac{2}{3}b^3}$$

$$\tau_{xy} = \frac{P}{2I} (b^2 - y^2)$$

$$\sigma_x = \frac{My}{I}$$



Variation of σ_x

Shear stress (τ_{xy})
varies parabolically

Note
for problems

- ① A cantilever of 2m length & cross section 80x120 mm subjected to load 1kN. calculate stress distribution at free end

Solution

$$P = 1000 \text{ N}$$

$$b = 80 \text{ mm}$$

$$l = 2 \text{ m}$$

$$h = 120 \text{ mm}$$

Refer
Mean Note

$$M = Pl = 1000 \times 2 = 2000 \text{ Nm}$$

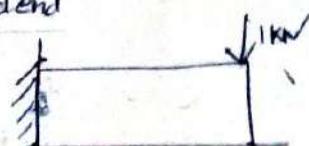
$$I = \frac{bh^3}{12} = \frac{80 \times 120^3}{12} = 1152 \times 10^8 \text{ mm}^4$$

$$\sigma_x = \frac{My}{I} = \frac{2000}{1152 \times 10^8} \times y \times 10^{-3} \text{ MPa}$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{P}{2I} (b^2 - y^2)$$

$$= -1 \times 10^{-3} (60^2 - y^2) \times 10^{-6}$$



Find 30, 40

y	0	10	20	30
bx	0	173	3472	
by	0	159	0.159	0.138