

Module-I

Introduction to stress Analysis's

Vectors, Scalar and Tensors

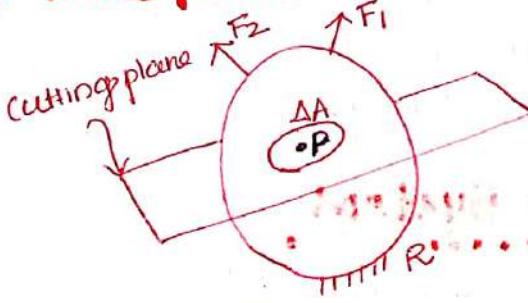
- Any physical quantity, which depends on both magnitude and direction known as vectors Eg: Force, Acceleration
- while Scalar are physical quantities depends on magnitude only

A Tensor is a multi-dimensional array of numerical value that can be used to describe the physical quantity Eg: Stress, Strain

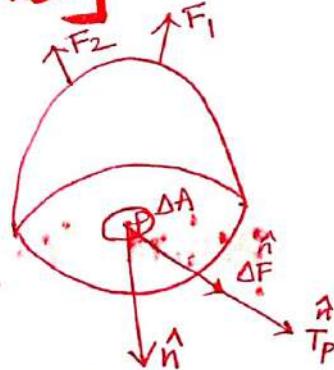
Stress Analysis \Rightarrow

Task of determine the stress field in an elastic solid called Stress analysis

Stress at a point [Resisting Traction]



Fig(a)



Fig(b)

Fig(a) Solid under equilibrium of external forces \vec{F}_1 and \vec{F}_2 . Body supported at bottom. Support reaction R

Eg: of static equilibrium

$$\vec{F}_1 + \vec{F}_2 + \vec{R} = 0$$

- Cutting plane cut body into two section

Traction at a point on plane

Fig(b) Consider one section P is point on solid ΔA small area around P

$\hat{\Delta F}$ = Resisting force over ΔA
 \hat{n} = unit normal

$$\frac{\hat{\Delta F}}{\hat{T}_P} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A}$$

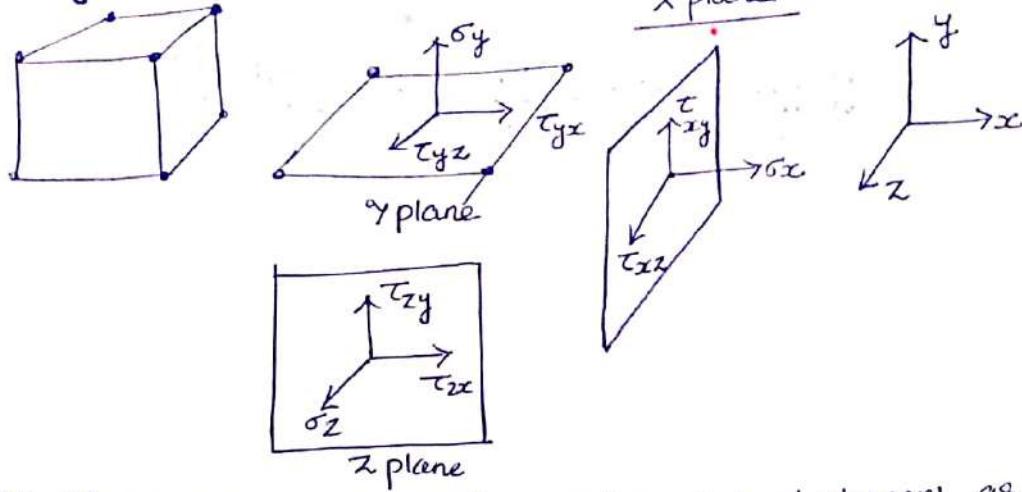
Stress Tensor [State of Stress at a point]

Definition: Infinite no' of plane passing through a point. Traction on each plane can be taken as Vector component of another. The highest order physical quantity called Stress Tensor (σ)

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}_{3 \times 3}$$

\Rightarrow 3D Stress Tensor
 $\sigma_{xx} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$
 $\sigma_{yy} = \frac{\sigma_{22} + \sigma_{33} - \sigma_{11}}{3}$
 $\sigma_{zz} = \frac{\sigma_{11} - \sigma_{22} - \sigma_{33}}{3}$

Rectangular Stress Tensor



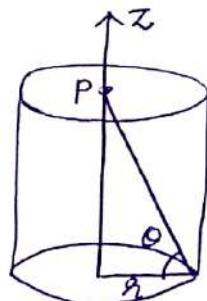
- Consider Three planes x , y and z with a cubical element as shown in figure.
- σ_x , σ_y and σ_z are the normal stress components acting \perp to x , y , z direction. Similarly, τ_{xy} , τ_{yz} ... etc are tangential stress acting on each plane. Then the rectangular stress component for the solid body is expressed in the Matrix in the form.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \Rightarrow \text{It is called Stress Tensor}$$

Polar Co-ordinate System

In polar co-ordinate system x , y , z in rectangular system can be replaced by r , θ , z respectively.

polar stress tensor expressed as $\vec{\sigma}_P = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{\theta z} & \sigma_z \end{bmatrix}$



Stress Transformation, Relation Connecting Cartesian and polar form

process of converting stress matrix of one co-ordinate system to another co-ordinate system is called Stress Transformation

→ consider a point P at a distance r from origin

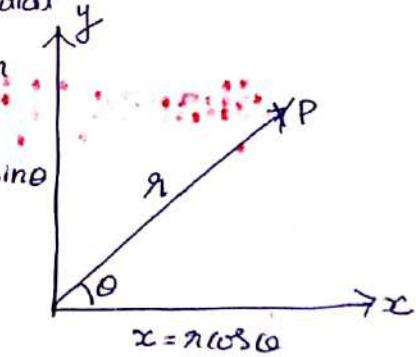
→ θ angle made with P w.r.t. z axis

$$x = r \cos \theta \quad y = r \sin \theta$$

$$\begin{aligned} \text{Vector form, } x\hat{i} + y\hat{j} + z\hat{k} &= r \cos \theta \hat{i} + r \sin \theta \hat{j} + 0 \hat{k} \\ &= r (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= r \hat{e}_\theta \end{aligned}$$

Similarly $\hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} + 0 \hat{k}$

$$\hat{e}_z = \hat{o}_i + \hat{o}_j + \hat{o}_k$$



Matrix Form

- $\hat{e}_r \rightarrow$ vector along radial direction
- $\hat{e}_\theta \rightarrow$ vector along tangential direction
- $\hat{e}_z \rightarrow$ " z direction

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\Omega]$$

Above Matrix is called Transformation Matrix

To convert any rectangular system into polar form, The relation as follows

$$[\sigma_p] = [\Omega] [\sigma_R] [\Omega]^T$$

$[\sigma_R]$ = Rectangular stress tensor

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ① The stress vectors on a plane x y z passing through a point are given below

$$\hat{T}_1 = 3i + 2j - 2k$$

(a) write down rectangular stress tensor

$$\hat{T}_2 = 2i - k + 0j$$

(b) Find transformation Matrix

$$\hat{T}_3 = -2i + 2k - j$$

(c) Evaluate polar stress tensor

by rotating the cartesian point through an angle 30°

(a)

$$[\sigma]_R = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

(b)

$$\theta = 30^\circ \Rightarrow \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.5 & 0 \\ -0.5 & 0.86 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\Omega]$$

(c)

$$[\sigma]_P = [\Omega] [\sigma]_R [\Omega]^T = \begin{bmatrix} 0.86 & 0.5 & 0 \\ -0.5 & 0.86 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.86 & -0.5 & 0 \\ 0.5 & 0.86 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.93 & -0.31 & -2.22 \\ -0.31 & -0.97 & -0.14 \\ -2.22 & 0.14 & 1 \end{bmatrix}$$

- ② The resulting traction vectors on x, y, z plane passing through a point (1, 1, 2) is given below.

$$\hat{T}_r = i + 2j$$

(a) write Matrix Stress tensor

$$\hat{T}_\theta = 2i + 3j$$

(b) convert into polar form

$$\hat{T}_z = 0k$$

$$(a) [\sigma_R] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tan\alpha = \frac{y}{x} = \frac{1}{1} \\ \theta = 45^\circ$$

$$(b) \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\sigma_p] = [\Omega] [\sigma_R] [\Omega]^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

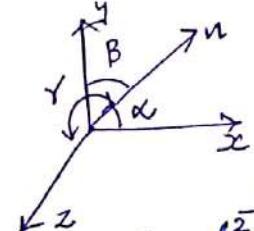
Unit Normal & direction cosines

- Unit Normal is any vector whose magnitude is one & its direction is perpendicular to a plane which is to be considered.

$$\vec{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$$

- Direction cosines are cosine of angle made by the normal vector with respect to all coordinate axis.

$$\boxed{\begin{aligned} n_x &= l = \cos \alpha \\ n_y &= m = \cos \beta \\ n_z &= n = \cos \gamma \end{aligned}}$$



Proof

$$l^2 + m^2 + n^2 = 1$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\alpha = \beta = \gamma$$

$$3 \cos^2 \alpha = 1 \Rightarrow \cos^2 \alpha = \frac{1}{3}$$

$$\cos \alpha = \frac{1}{\sqrt{3}}$$

- Find direction cosines of following cases.

(a) A plane equally inclined to 3 axis Ans: $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$

(b) A line with equation $4x + 2y - z = 1$ Normal vector $\rightarrow 4\hat{i} + 2\hat{j} - \hat{k}$

$$\text{Ans: } n_x = \frac{4}{\sqrt{4^2 + 2^2 + 1^2}} = 0.87, n_y = \frac{2}{\sqrt{4^2 + 2^2 + 1^2}} = 0.43, n_z = \frac{-1}{\sqrt{4^2 + 2^2 + 1^2}} = -0.21$$

(c) A plane passing through point $(1, 2, 1)$, $(0, 0, 0)$, $(1, 1, 0)$

A B C

$$\vec{AB} = \vec{B} - \vec{A} = -\hat{i} - 2\hat{j} - \hat{k}$$

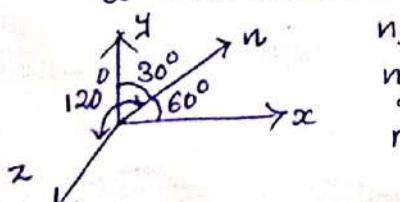
$$\vec{BC} = \vec{C} - \vec{B} = \hat{i} + \hat{j} + 0\hat{k}$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}(1) - \hat{j}(1) + \hat{k}(1) = \hat{i} - \hat{j} + \hat{k}$$

$$\frac{|\vec{AB} \times \vec{BC}|}{|\vec{AB} \times \vec{BC}|} = \frac{\sqrt{1^2 + 1^2 + 1^2}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{\sqrt{3}}{\sqrt{3}} = 1$$

$$\left\{ \begin{aligned} n_x &= \frac{1}{\sqrt{3}} \\ n_y &= -\frac{1}{\sqrt{3}} \\ n_z &= \frac{1}{\sqrt{3}} \end{aligned} \right.$$

- (d) A plane Normal Vector make an angle 60° with x axis



$$n_x = \cos 60^\circ$$

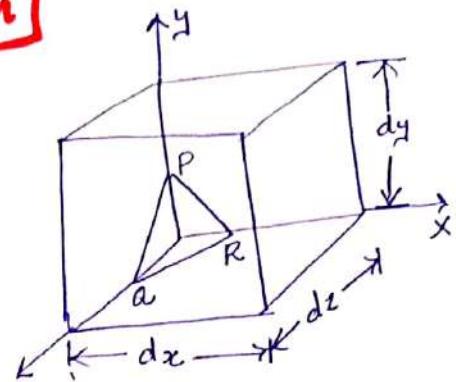
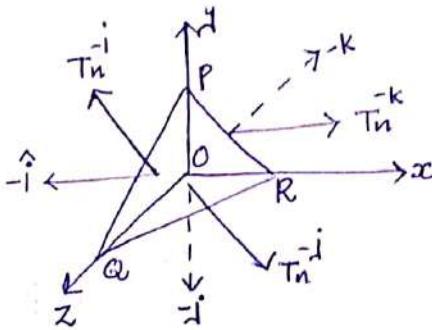
$$n_y = \cos 30^\circ$$

$$n_z = \cos 120^\circ$$

Stress on arbitrary plane

(3)

[Cauchy's Equation]



If the stress component acting along any 3 mutually perpendicular plane passing through a point are given, stresses acting along any arbitrary plane passing through point can be determined.

→ Consider small tetrahedron OPR cut from cubic element as shown in fig.

$$\begin{cases} \text{Area of } PQR = \Delta A \\ \text{Area of } OQR = n_y \Delta A \\ \text{Area of } OPL = n_x \Delta A \\ \text{Area of } OPR = n_z \Delta A \end{cases}$$

$$\begin{aligned} \text{Traction} &= \frac{\text{Load}}{\text{Area}} \\ \text{Load} &= \text{Traction} \times \text{Area} \end{aligned}$$

⇒ For Equilibrium sum of all total load will be zero

$$T_n \Delta A + T_n^-i n_x \Delta A + T_n^-j n_y \Delta A + T_n^-k n_z \Delta A + mg = 0$$

Smaller tetrahedron weight can be neglected ⇒ $mg = 0$

$$T_n \Delta A + T_n^-i n_x \Delta A + T_n^-j n_y \Delta A + T_n^-k n_z \Delta A = 0$$

$$T_n \Delta A - T_n^-i n_x \Delta A - T_n^-j n_y \Delta A - T_n^-k n_z \Delta A = 0$$

$$T_n = T_n^i n_x + T_n^j n_y + T_n^k n_z \quad \text{--- (1)}$$

Vector properties

$$\begin{aligned} T_n^i &= -T_i \\ T_n^-j &= -T_j \\ T_n^-k &= -T_k \end{aligned}$$

$$\Rightarrow T_n = (\sigma_x i\hat{+} \tau_{xy} j\hat{+} \tau_{xz} k\hat{)} n_x + (\tau_{yx} i\hat{+} \sigma_y j\hat{+} \tau_{yz} k\hat{)} n_y + (\tau_{zx} i\hat{+} \tau_{zy} j\hat{+} \sigma_z k\hat{)} n_z$$

$$\Rightarrow T_n = (\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z) \hat{i} + (\tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z) \hat{j} + (\tau_{zx} n_z + \tau_{zy} n_y + \sigma_z n_z) \hat{k}$$

Component of traction in x, y, z are

These are called
Cauchy's
Equation

$$\begin{aligned} T_x^n &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ T_y^n &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ T_z^n &= \tau_{xz} n_z + \tau_{yz} n_y + \sigma_z n_z \end{aligned}$$

Cauchy's Eq
Matrix Form

$$\begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

Stress Matrix

Direction cosines

Q. No. 1 Equation For Normal Stress & Shear Stress

$$|T_n|^2 = \sigma_n^2 + \tau_n^2 \rightarrow (a) \quad \left\{ \begin{array}{l} \sigma_n - \text{Normal Stress} \\ \tau_n - \text{Shear Stress} \end{array} \right.$$

$$\boxed{\sigma_n = (T_x^n \hat{i} + T_y^n \hat{j} + T_z^n \hat{k}) \cdot (n_x \hat{i} + n_y \hat{j} + n_z \hat{k}) \rightarrow (b)} \\ = T_x^n n_x + T_y^n n_y + T_z^n n_z$$

From (a) & (b) Value of τ_n calculated

- (1) The components of stress matrix are follows $\sigma_{xx}=5, \sigma_{yy}=3, \sigma_{zz}=2, \tau_{yx}=2, \tau_{yz}=1, \tau_{xz}=1$. Determine the stress on a plane equally inclined to 3 axis.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{bmatrix} \quad \text{where } n_x = n_y = n_z = \frac{1}{\sqrt{3}} \quad (\text{equally inclined 3 axis})$$

$$\begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 4.61 \\ 5.19 \\ 4.09 \end{bmatrix}$$

$$\boxed{|T_n|^2 = \sigma_n^2 + \tau_n^2 \rightarrow (1)} \Rightarrow \sigma_n = (T_x^n \hat{i} + T_y^n \hat{j} + T_z^n \hat{k}) \cdot (n_x \hat{i} + n_y \hat{j} + n_z \hat{k}) \\ = (4.61 \hat{i} + 5.19 \hat{j} + 4.09 \hat{k}) \cdot \left(\frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \right)$$

$$\sigma_n = 2.661 + 2.99 + 2.33 = \underline{\underline{7.98}}$$

$$\{ T_n = T_x^n \hat{i} + T_y^n \hat{j} + T_z^n \hat{k} \} \Rightarrow |T_n|^2 = 4.61^2 + 5.19^2 + 4.09^2 = \underline{\underline{64.50}}$$

Sub: value of σ_n & $|T_n|^2$ in eq - (1) $64.50^2 = 7.98^2 + \tau_n^2$

$$\tau_n = \sqrt{64.50^2 - 7.98^2} = \underline{\underline{0.89 \text{ MPa}}}$$

- (2) The stress at a point is given by following matrix $\sigma_y=20, \sigma_z=30, \tau_{yz}=-20, \tau_{yz}=10, \tau_{zx}=40$ and $\sigma_x=50$. Determine the stress on plane whose direction cosines are $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$

Solution

$$n_x = \frac{1}{\sqrt{2}}, n_y = \frac{1}{2}, n_z = \frac{1}{2}$$

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 50 & -20 & 40 \\ -20 & 20 & 10 \\ 40 & 10 & 30 \end{bmatrix}$$

$$\begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 50 & -20 & 40 \\ -20 & 20 & 10 \\ 40 & 10 & 30 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 45.355 \\ 0.85 \\ 48.284 \end{bmatrix}$$

$$|T_n|^2 = \sigma_n^2 + \tau_n^2 \rightarrow (1)$$

$$\sigma_n = (T_x^n \hat{i} + T_y^n \hat{j} + T_z^n \hat{k}) \cdot (n_x \hat{i} + n_y \hat{j} + n_z \hat{k}) = (45.355 \hat{i} + 0.85 \hat{j} + 48.284 \hat{k}) \cdot \left(\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{2} \hat{j} + \frac{1}{2} \hat{k} \right) \\ = \underline{\underline{56.637}}$$

$$|T_{n1}|^2 = 45.355^2 + 0.85^2 + 48.284^2 = \underline{4389.14}$$

$$|\tau_n|^2 = \sigma_n^2 + \tau_n^2$$

$$\tau_n^2 = \sqrt{|T_{n1}|^2 - \sigma_n^2} = \sqrt{4389.14 - 3207.74} = \underline{34.37 \text{ Pa}}$$

- ③ At a point in a stressed material component of stress are $\sigma_x = 40, \sigma_y = 30$, $\sigma_z = 120, \tau_{xz} = 82, \tau_{yz} = 46, \tau_{xy} = 72$

(a) Obtain Stress tensor

(b) Find direction cosines

(c) Find magnitude of traction vector

(d) Calculate normal & shear component on a plane in which normal make an angle 48° with x , 61 with y axis, 90 with z axis

Solution

$$(a) \sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 40 & 72 & 82 \\ 72 & 30 & 46 \\ 82 & 46 & 120 \end{bmatrix}$$

$$(b) n_x = \cos \alpha = \cos 48 = 0.66$$

$$n_y = \cos \beta = \cos 61 = 0.48$$

$$n_z = \cos \gamma = \cos 90 = 0$$

$$(c) \begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} 40 & 72 & 82 \\ 72 & 30 & 46 \\ 82 & 46 & 120 \end{bmatrix} \begin{bmatrix} 0.66 \\ 0.48 \\ 0 \end{bmatrix} = \begin{bmatrix} 60.96 \\ 60.6 \\ 43.2 \end{bmatrix} \Rightarrow T_n = 60.96\hat{i} + 60.6\hat{j} + 43.2\hat{k}$$

$$|T_n|^2 = 60.96^2 + 60.6^2 + 43.2^2 = \underline{9254.7216}$$

$$|T_{n1}|^2 = \sigma_n^2 + \tau_n^2 \quad (1) \quad \& \quad \sigma_n = T_n \cdot \hat{n} = (60.96\hat{i} + 60.6\hat{j} + 43.2\hat{k}) (0.66\hat{i} + 0.48\hat{j} + 0\hat{k}) \\ = 40.233 + 29.088 + 0 = \underline{69.321}$$

$$\tau_n^2 = \sqrt{|T_{n1}|^2 - \sigma_n^2} = \sqrt{9254.7216 - 69.321^2} = \underline{66.703 \text{ Pa}}$$

- ④ State of stress at a point is $[\sigma_{ij}] = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Determine normal & shear stresses on the plane equally inclined to all three axes.

A plane equally inclined to 3 axes, $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$

$$\begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{6}{\sqrt{3}} \\ \frac{6}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{bmatrix} \Rightarrow \vec{T} = \frac{6}{\sqrt{3}}\hat{i} + \frac{6}{\sqrt{3}}\hat{j} + \frac{2}{\sqrt{3}}\hat{k}$$

$$\sigma_n = \vec{T} \cdot \vec{n} = \left(\frac{6}{\sqrt{3}}\hat{i} + \frac{6}{\sqrt{3}}\hat{j} + \frac{2}{\sqrt{3}}\hat{k} \right) \cdot \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) = \underline{\frac{14}{3}}$$

$$|T_n|^2 = \sigma_n^2 + \tau_n^2$$

$$T_n = \sqrt{|T_n|^2 - \sigma_n^2} = \sqrt{\left(\frac{14}{3}\right)^2 - \left(\frac{14}{3}\right)^2} = \underline{1886 \text{ unit}} = \sqrt{\frac{14}{3}}$$

$$|T_n| = \sqrt{\left(\frac{6}{\sqrt{3}}\right)^2 + \left(\frac{6}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \sqrt{\frac{76}{3}}$$

- ⑤ An axially loaded square shaped bar subjected to an average force of $F = 10\text{N}$ shown in f.g. The cross sectional area is 10mm^2

- (a) Write down stress tensor. For a plane parallel to z axis and inclined 60° with x-axis
 (b) Resisting traction σ magnitude
 (c) Normal stress σ Normal stress vector
 (d) Shear stress vector τ magnitude

$$\text{Here } n_x = \cos \alpha = \cos 30 = \frac{\sqrt{3}}{2}$$

$$n_y = \cos \beta = \cos 120 = -0.5$$

$$(n_z = \cos 270), n_z = \cos 90 = 0$$

(a) Stresses only one component (Load in x direction only)

$$\sigma_x = F/A = \frac{10}{10} = 1 \text{ N/mm}^2$$

Stress tensor $[\sigma] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$(b) \begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_n = \vec{T} \cdot \vec{n} = \left(\frac{\sqrt{3}}{2} \hat{i} + 0 \hat{j} + 0 \hat{k} \right) \cdot \left(\frac{\sqrt{3}}{2} \hat{i} + (-0.5) \hat{j} + 0 \hat{k} \right)$$

$$= \frac{3}{4} \text{ N/mm}^2$$

$$|\vec{T}|^2 = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + 0^2 + 0^2} = \frac{3}{4}$$

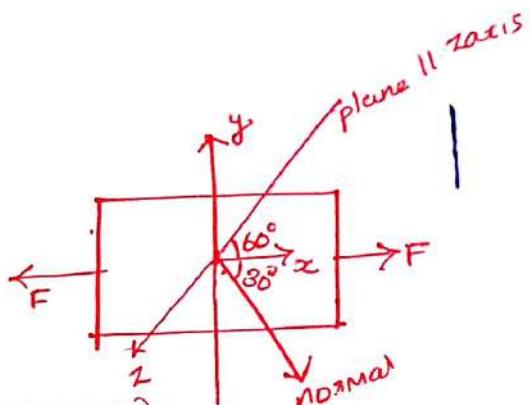
$$(c) |\vec{T}|^2 = \sigma_n^2 + T_n^2 \Rightarrow T_n = \sqrt{|\vec{T}|^2 - \sigma_n^2} = \sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{3}}{4} \text{ N/mm}^2$$

Principal Plane & Principal Stress

- * By Cauchy's formula we can determine normal and shear stresses component acting along any plane through a point.
- * There exist a plane or plane only normal stress acting and shear stresses are zero. Such plane are called principal plane or shearless plane and normal stress acting on this plane are called principal stress.

By Cauchy's Formula

$$\begin{bmatrix} T_x^n \\ T_y^n \\ T_z^n \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \quad \text{--- (1)}$$



(5)

By concept of principal stress

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \quad \text{--- (2)}$$

Equate eq. (1) and (2)

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

eq. (3)

To above eq. n_x, n_y and n_z cannot be zero
In order to obtain non-zero solution, the value of determinant of fig.
st matrix must be zero.

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma - \sigma \end{vmatrix} = 0$$

on expanding we get $\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + \left[\begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} \right] \sigma$

$$- \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = 0$$

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \quad \text{--- (A)}$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z$, $I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix}$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}, \text{ where } I_1, I_2 \text{ and } I_3 \text{ are called stress invariants}$$

Note Stress invariants are related to stress tensor which do not change with rotation of axis

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \rightarrow \text{is called characteristics eq. of principal stress, root are } \sigma_1, \sigma_2, \sigma_3$$

σ_1 = Largest one [Major principal stress]

σ_2 = Second largest

σ_3 = Third largest [Minor principal stress]

Q. no.

$$\boxed{\text{Max shear stress} \quad \tau_{max} = \frac{\sigma_1 - \sigma_3}{2}}$$

- ① The state of stress at a point is given by Cartesian stress tensor
 $\sigma_x = 3, \sigma_y = 5, \sigma_z = 3$ & $\tau_{xy} = -1, \tau_{xz} = 1, \tau_{yz} = -1$ (a) Obtain stress tensor
 (b) Find 3 stress invariants
 (c) Find principal stresses
 (d) Check invariance

Solution

(a) Stress tensor $[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(b) $I_1 = \sigma_x + \sigma_y + \sigma_z = 3 + 5 + 3 = 11$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{yz} \\ \tau_{xy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yz} & \sigma_y \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 15 - (1) + 9 - 1 + 15 - 1 = \underline{\underline{36}}$$

(c) $\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \Rightarrow \sigma^3 - 11\sigma^2 + 36\sigma - 36 = 0$

$$I_3 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = \underline{\underline{36}}$$

(d) $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 6 + 3 + 2 = \underline{\underline{11}}, I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = 6 \times 3 + 3 \times 2 + 2 \times 6 = \underline{\underline{36}}$
 $I_3 = \sigma_1\sigma_2\sigma_3 = 6 \times 3 \times 2 = \underline{\underline{36}}$

- ② The state of stress at a point given by $\sigma_x = 70, \sigma_y = 10, \sigma_z = -20$ (N/mm²)
 $\tau_{xy} = -40 \text{ N/mm}^2, \tau_{zy} = \tau_{xz} = 20 \text{ N/mm}^2$

(a) Determine stress invariance

(b) Find principal stresses

(c) Find direction of max shear stress

(d) Find Max shear stress

Solution

(a) $[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 70 & -40 & 20 \\ -40 & 10 & 20 \\ 20 & 20 & -20 \end{bmatrix}$

$I_1 = \sigma_x + \sigma_y + \sigma_z = 70 + 10 - 20 = \underline{\underline{60}}$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{yz} \\ \tau_{xy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yz} & \sigma_y \end{vmatrix} = \begin{vmatrix} 10 & 20 \\ 20 & -20 \end{vmatrix} + \begin{vmatrix} 70 & 20 \\ 20 & -20 \end{vmatrix} + \begin{vmatrix} 70 - 40 & 20 \\ -40 & 10 \end{vmatrix}$$

$$= \underline{\underline{-3300}}$$

$I_3 = \begin{vmatrix} 70 - 40 & 20 \\ -40 & 10 & 20 \\ 20 & 20 & -20 \end{vmatrix} = \underline{\underline{-46000}}$

$$\textcircled{b} \quad \sigma^3 - 60\sigma^2 + 3300\sigma + 46000 = 0 \quad [\sigma^3 - I_1\sigma^2 + I_2\sigma + I_3 = 0] \quad \textcircled{d}$$

$$\sigma_1 = 90.77, \sigma_2 = 118, \sigma_3 = -426$$

⑥ Find direction of any principal stress Subtract 1st diagonal element

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 70 - 90.77 & -40 & 20 \\ -40 & 10 - 90.77 & 20 \\ 20 & 20 & -20 - 90.77 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-20.77n_x - 40n_y + 20n_z = 0 \quad \textcircled{1}$$

$$-40n_x - 80.77n_y + 20n_z = 0 \quad \textcircled{2}$$

$$20n_x + 20n_y - 110.77n_z = 0 \quad \textcircled{3}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow -20.77n_x - 40n_y + 20n_z = 0 \quad \textcircled{1}$$

$$-40n_x - 80.77n_y + 20n_z = 0 \quad \textcircled{2}$$

$$19.23n_x + 40.77n_y$$

Similarly

$$\textcircled{2} - \textcircled{3} \Rightarrow -20.77n_x - 40n_y + 20n_z = 0$$

$$40n_x + 40n_y - 221.54n_z = 0$$

$$19.23n_x + 201.54n_z$$

$$n_z = 0.095n_x$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_x^2 + (-0.472n_x)^2 + (0.095n_x)^2 = 1$$

$$n_x = 0.901, n_y = -0.424, n_z = 0.085$$

$$\text{Max shear stress} = \frac{90.77 - (-42.6)}{2} = 66 \text{ MPa}$$

$$\frac{\sigma_1 - \sigma_3}{2}$$

③ The state of stress at a point is characterised by $\sigma_x = 18, \sigma_y = -50, \sigma_z = 32$, $\tau_{xy} = 0, \tau_{xz} = 24, \tau_{yz} = 0$ (kPa). Calculate principal stresses and direction of largest tensile principal stress. Also determine Max shear stress & check invariance

Solution

$$\text{Stress tensor } [\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{bmatrix} \text{ kPa}$$

$$\text{Stress invariants } I_1 = \sigma_x + \sigma_y + \sigma_z = 18 - 50 + 32 = 0$$

$$I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} = \begin{vmatrix} -50 & 0 \\ 0 & 32 \end{vmatrix} + \begin{vmatrix} 18 & 24 \\ 24 & 32 \end{vmatrix} + \begin{vmatrix} 18 & 0 \\ 0 & -50 \end{vmatrix}$$

$$= -2500$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = \begin{vmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{vmatrix} = 0$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \Rightarrow \sigma^3 - 0 - 2500\sigma - 0 = 0$$

Principal Stress

$$\sigma_1 = 50 \text{ kPa}, \sigma_2 = 0, \sigma_3 = -50 \text{ kPa}$$

Direction of largest principal stress $[\sigma_1 = 50] = 5$

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 18-50 & 0 & 24 \\ 0 & -50-50 & 0 \\ 24 & 0 & 32-50 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$n_x = \begin{vmatrix} 0 & -100 \\ 24 & 0 \end{vmatrix} = 0 - (-100 \times 24) = \underline{\underline{2400}}, n_z = \begin{vmatrix} -100 & 0 \\ 0 & -18 \end{vmatrix} = \underline{\underline{1800}}$$

$$n_{xz} = \begin{vmatrix} 0 & 0 \\ 24 & -18 \end{vmatrix} = 0$$

$$n_x = \frac{2400}{\sqrt{2400^2 + 1800^2 + 0}} = 0.6, n_y = 0, n_z = \frac{1800}{\sqrt{2400^2 + 1800^2 + 0}} = 0.8$$

$$\text{Max shear stress} = \frac{\sigma_1 - \sigma_3}{2} = \frac{50 - (-50)}{2} = \frac{100}{2} = \underline{\underline{50 \text{ kPa}}}$$

or

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 18-50 & 0 & 24 \\ 0 & -50-50 & 0 \\ 24 & 0 & 32-50 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -32n_x + 0n_y + 24n_z &= 0 & \text{eq-1} \\ 0n_x - 100n_y + 0n_z &= 0 & \text{eq-2} \\ 24n_x + 0n_y - 18n_z &= 0 & \text{eq-3} \end{aligned}$$

$$\text{From eq-2 } n_y = 0$$

$$\text{eq-1 } -32n_x + 24n_z = 0$$

$$\text{eq-3 } 24n_x - 18n_z = 0$$

$$\text{eq-3 } \frac{24n_x = 18n_z}{n_z = 1.33n_x} \quad n_x^2 + n_y^2 + n_z^2 = 1$$

$$\Rightarrow n_x^2 + 0 + (1.33n_x)^2 = 1 \Rightarrow n_x = 0.6, n_z = 1.33 \times 0.6 = \underline{\underline{0.8}}$$

Hydrostatic & deviatoric stresses

The state of stress at a point decomposed into two parts, Hydrostatic and deviatoric parts.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \underbrace{\begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix}}_{\text{Hydrostatic Stress Matrix}} + \underbrace{\begin{bmatrix} \sigma_x - P & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - P & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - P \end{bmatrix}}_{\text{Deviatoric Stress Component}}$$

- It having same value of principal stress $= P$

$$\sigma_x = \sigma_y = \sigma_z = P \quad \tau_{xy} = \tau_{xz} = \tau_{yz} = \tau_{zx} = \tau_{zy} = 0$$

Normal stress $= P$ $\tau_{xy} = 0$ (shear stresses = 0)

This characteristic identical to the hydrostatic pressure acting at a point in a static fluid. Hence it is called Hydrostatic stress.

Here change volume without shape change

• Harmless

First invariants

$$I_1 = \sigma_x - P + \sigma_y - P + \sigma_z - P$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z - 3P \Rightarrow P = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

When a body is acted upon by 3 system of stresses, deviatoric part produces all change of shape in the body.

where $I_1 = 0 \Rightarrow$ deviatoric state is called State of pure shear

① Decompose the given stress matrix into hydrostatic and deviatoric part (9)

$$\sigma_x = 51, \sigma_y = 42, \sigma_z = 50, \tau_{xy} = 0, \tau_{xz} = 24, \tau_{yz} = 0$$

Solution

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} \sigma_x - P & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - P & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - P \end{bmatrix}$$

$$\text{where } P = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \\ = \frac{51 + 50 + 42}{3} = 47.66$$

$$\begin{bmatrix} 51 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 42 \end{bmatrix} = \begin{bmatrix} 47.66 & 0 & 0 \\ 0 & 47.66 & 0 \\ 0 & 0 & 47.66 \end{bmatrix} + \begin{bmatrix} 3.34 & 0 & 24 \\ 0 & 2.34 & 0 \\ 24 & 0 & -5.66 \end{bmatrix}$$

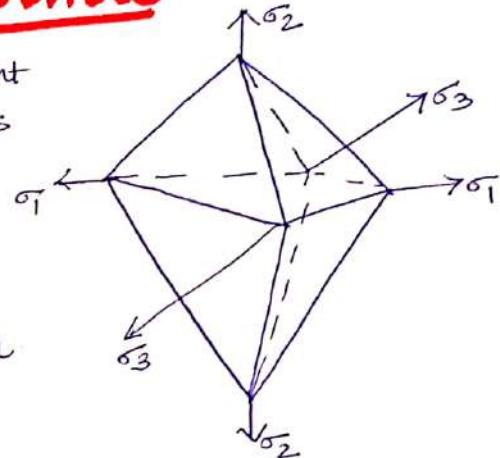
Hydrostatic

Deviatoric

Octahedral Stress

- An octahedron is a solid having eight identical equilateral triangles. Any of this plane is called octahedral plane and stresses acting on this plane is called Octahedral stresses.
- All triangles are equally inclined to reference plane for such plane direction cosines are equal:

$$n_x = n_y = n_z = \frac{1}{\sqrt{3}}$$



Cauchy's eq in terms of principal stress is

$$\text{Here } n_x = n_y = n_z = \frac{1}{\sqrt{3}}$$

$$\begin{bmatrix} \tau_x^n \\ \tau_y^n \\ \tau_z^n \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \Rightarrow \begin{bmatrix} \tau_x^n \\ \tau_y^n \\ \tau_z^n \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \sigma_x/\sqrt{3} \\ \sigma_y/\sqrt{3} \\ \sigma_z/\sqrt{3} \end{bmatrix}$$

$$\vec{T} = \frac{\sigma_x}{\sqrt{3}} \hat{i} + \frac{\sigma_y}{\sqrt{3}} \hat{j} + \frac{\sigma_z}{\sqrt{3}} \hat{k} \Rightarrow |\vec{T}| = \sqrt{\left(\frac{\sigma_x}{\sqrt{3}}\right)^2 + \left(\frac{\sigma_y}{\sqrt{3}}\right)^2 + \left(\frac{\sigma_z}{\sqrt{3}}\right)^2} \Rightarrow |\vec{T}|^2 = \frac{\sigma_x^2}{3} + \frac{\sigma_y^2}{3} + \frac{\sigma_z^2}{3}$$

$$\vec{n} = \frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k}$$

$$\text{Normal stress} \Rightarrow \sigma_n = \vec{T} \cdot \vec{n} = \left(\frac{\sigma_x}{\sqrt{3}} \hat{i} + \frac{\sigma_y}{\sqrt{3}} \hat{j} + \frac{\sigma_z}{\sqrt{3}} \hat{k} \right) \cdot \left(\frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \right)$$

$$\sigma_{\text{oct}} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \Rightarrow \sigma_{\text{oct}} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$\text{we have } |\vec{T}|^2 = \sigma_n^2 + \tau_n^2 = \sigma_{\text{oct}}^2 + \tau_{\text{oct}}^2 \quad \text{--- (2)}$$

$$\frac{\sigma_x^2}{3} + \frac{\sigma_y^2}{3} + \frac{\sigma_z^2}{3} = \left(\frac{\sigma_x + \sigma_y + \sigma_z}{3} \right)^2 + \tau_{\text{oct}}^2 \Rightarrow \tau_{\text{oct}}^2 = \left(\frac{\sigma_x + \sigma_y + \sigma_z}{3} \right)^2 - \frac{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}{3}$$

$$\tau_{\text{oct}}^2 = \frac{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}{3} - \left(\frac{\sigma_x + \sigma_y + \sigma_z}{3} \right)^2 \Rightarrow \tau_{\text{oct}}^2 = \frac{1}{9} [3(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - (\sigma_x + \sigma_y + \sigma_z)^2]$$

$$\tau_{\text{oct}}^2 = \frac{1}{9} [3\sigma_x^2 + 3\sigma_y^2 + 3\sigma_z^2 - \sigma_x^2 - \sigma_y^2 - \sigma_z^2 - 2\sigma_x\sigma_y - 2\sigma_y\sigma_z - 2\sigma_z\sigma_x]$$

$$\tau_{\text{oct}}^2 = \frac{1}{9} [2\sigma_x^2 + 2\sigma_y^2 + 2\sigma_z^2 - 2\sigma_x\sigma_y - 2\sigma_y\sigma_z - 2\sigma_z\sigma_x]$$

$$\tau_{oct}^2 = \frac{2}{9} \left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x \right]$$

It can also be written as

$$\tau_{oct}^2 = \frac{2}{9} \left[(\underbrace{\sigma_x + \sigma_y + \sigma_z}_I)^2 - 3(\underbrace{\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x}_J) \right]$$

$$\tau_{oct}^2 = \frac{2}{9} [I_1^2 - 3I_2] \Rightarrow \boxed{\tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{I_1^2 - 3I_2}} \quad \begin{array}{l} \text{Octahedral} \\ \text{Shear stress} \end{array}$$

① The state of Stress at a point given by

$$\begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 42 \end{bmatrix}$$

- (a) What is octahedral normal & shear stress Hydrostatic & deviatoric part
 (b) First & Second invariance
 (c) Decompose into hydrostatic & deviatoric part

(b) $[\sigma] = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 42 \end{bmatrix} \Rightarrow I_1 = \sigma_x + \sigma_y + \sigma_z = 57 + 50 + 42 = \underline{149}$
 $I_2 = \begin{vmatrix} 50 & 0 \\ 0 & 42 \end{vmatrix} + \begin{vmatrix} 57 & 24 \\ 24 & 42 \end{vmatrix} + \begin{vmatrix} 57 & 0 \\ 0 & 50 \end{vmatrix} = \underline{6768}$

(c) $\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} \sigma_x - P & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - P & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - P \end{bmatrix}$ where $P = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{57 + 50 + 42}{2} = \underline{49.66}$

$$\begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 42 \end{bmatrix} = \underbrace{\begin{bmatrix} 49.66 & 0 & 0 \\ 0 & 49.66 & 0 \\ 0 & 0 & 49.66 \end{bmatrix}}_{\text{Hydrostatic}} + \underbrace{\begin{bmatrix} 7.34 & 0 & 24 \\ 0 & 0.34 & 0 \\ 0 & 0 & -7.66 \end{bmatrix}}_{\text{Deviatoric}}$$

(d) Hydrostatic

$$\sigma_{oct} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{49.66 + 49.66}{3} = \underline{49.66}$$

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{I_1^2 - 3I_2}$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 49.66 + 49.66 + 49.66 = \underline{148.98}$$

$$I_2 = \begin{vmatrix} 49.66 & 0 \\ 0 & 49.66 \end{vmatrix} + \begin{vmatrix} 49.66 & 0 \\ 0 & 49.66 \end{vmatrix} + \begin{vmatrix} 49.66 & 0 \\ 0 & 49.66 \end{vmatrix} = \underline{7398.34}$$

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{148.98^2 - 3 \times 7398.34} = \underline{0.06} \text{ (Approximately)}$$

Deviatoric

$$\sigma_{oct} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{7.34 + 0.34 - 7.66}{3} = \underline{0.002}$$

$$I_1 = 7.34 + 0.34 - 7.66 = \underline{0.02}$$

$$I_2 = \begin{vmatrix} 0.34 & 0 \\ 0 & -7.66 \end{vmatrix} + \begin{vmatrix} 7.34 & 24 \\ 0 & -7.66 \end{vmatrix} + \begin{vmatrix} 7.34 & 0 \\ 0 & 0.34 \end{vmatrix} = \underline{-632.32}$$

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{(0.02)^2 - 3 \times -632.32} = \underline{20.53}$$

H.W ② Given the state of stress at a point $\sigma_x = 50, \sigma_y = 20, \sigma_z = -20, \tau_{xz} = 0$, $\tau_{yz} = 80, \tau_{xy} = 50$ ③

- (a) Find Hydrostatic component of Stress tensor
- (b) Find octahedral shear stress of hydrostatic part
- (c) Third Stress invariance

H.W ③ The state of stress of a point is given by $\sigma_x = 100, \sigma_y = -40, \sigma_z = 80$ & all other element are zero. Determine the octahedral shear stress & associated normal stress of hydrostatic part

Solution

③ (a) $\sigma = \begin{bmatrix} 100 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & 80 \end{bmatrix}$ where $P = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{100 - 40 + 80}{3} = \underline{\underline{16.66}}$

Hydrostatic part $\Rightarrow \begin{bmatrix} 16.66 & 0 & 0 \\ 0 & 16.66 & 0 \\ 0 & 0 & 16.66 \end{bmatrix}$

(b) $\tau_{\text{oct}} = \frac{\sqrt{2}}{3} \sqrt{I_1^2 - 3I_2}$

$$\tau_{\text{oct}} = \frac{\sqrt{2}}{3} \sqrt{50^2 - 3 \times 4354.703} = 0 //$$

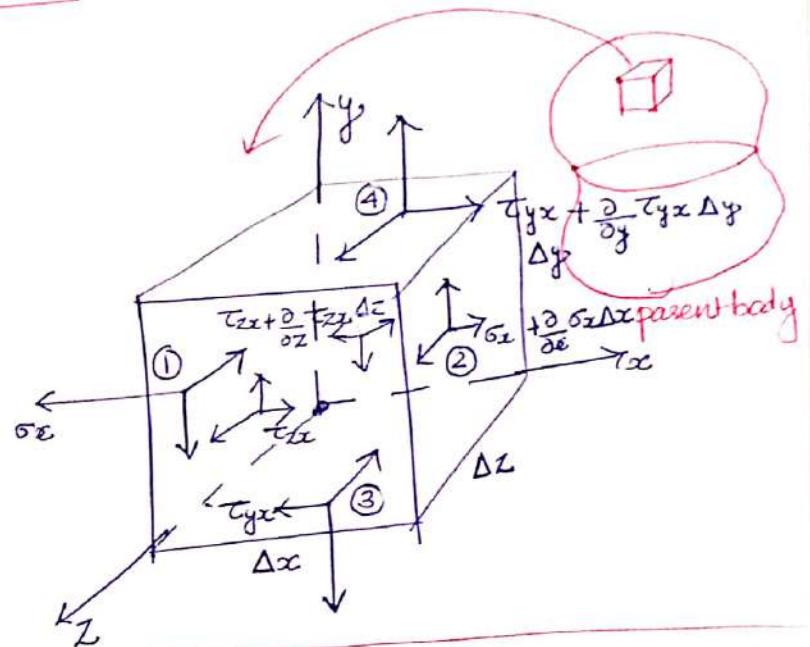
$$I_1 = 16.66 + 16.66 + 16.66 = 50$$

$$I_2 = \begin{vmatrix} 16.66 & 0 & 0 \\ 0 & 16.66 & 0 \\ 0 & 0 & 16.66 \end{vmatrix} + \begin{vmatrix} 16.66 & 0 & 0 \\ 0 & 16.66 & 0 \\ 0 & 0 & 16.66 \end{vmatrix} + \begin{vmatrix} 16.66 & 0 & 0 \\ 0 & 16.66 & 0 \\ 0 & 0 & 16.66 \end{vmatrix} = \underline{\underline{4354.703}}$$

(c) $I_3 = \begin{vmatrix} 100 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & 80 \end{vmatrix} = \underline{\underline{-290000}}$

Equilibrium Equation in Cartesian/Rectangular Coordinate System

- Equilibrium eq. obtained by considering very small volume of element from parent body
- Consider small rectangular element of side $\Delta x, \Delta y, \Delta z$
- Body has 6 faces



Faces	Average Stress Component	Faces	Average Stress Component
Left (1)	$\sigma_x, \tau_{xy}, \tau_{xz}$	Top (4)	$\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y, \sigma_y + \frac{\partial \sigma_y}{\partial y} \Delta y$ $\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} \Delta y$
Right (2)	$\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x, \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \Delta x$ $\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \Delta x$	Back (5)	$\tau_{zx}, \tau_{zy}, \sigma_z$
Bottom (3)	$\tau_{yx}, \sigma_y, \tau_{yz}$	Front (6)	$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z, \tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} \Delta z$ $\sigma_z + \frac{\partial \sigma_z}{\partial z} \Delta z$

At Equilibrium Sum of all forces in x direction = 0
[$\gamma_x, \gamma_y, \gamma_z$ = Body forces in x, y, z direction]

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x \right) \Delta y \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y \right) \Delta x \Delta z + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \right) \Delta x \Delta y - \sigma_x \Delta y \Delta z - \tau_{yz} \Delta x \Delta z - \tau_{zx} \Delta x \Delta y + \gamma_x \Delta x \Delta y \Delta z = 0$$

All term divided by $\Delta x \Delta y \Delta z$

$$\frac{\sigma_x}{\Delta x} + \frac{\partial \sigma_x}{\partial x} + \frac{\tau_{yx}}{\Delta y} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\tau_{zx}}{\Delta z} + \frac{\partial \tau_{zx}}{\partial z} - \frac{\sigma_x}{\Delta x} - \frac{\tau_{yz}}{\Delta y} - \frac{\tau_{zx}}{\Delta z} + \gamma_x = 0$$

x direction	$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial z} \tau_{zx} + \gamma_x = 0$ — (1)	Equilibrium equation [3D]
Similarly	$\frac{\partial \tau_{yz}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \gamma_y = 0$ — (2)	
y direction	$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \gamma_z = 0$ — (3)	

plane stress [2D] τ_{xy} only

$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \gamma_x = 0$	$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \gamma_y = 0$
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① The stress field of body given by $\sigma_x = 20x^2 + y^2$, $\sigma_y = 30x^3 + 200$
 $\sigma_z = 30(y^2 + z^2)$, $\tau_{xy} = \tau_{yz} = zx$, $\tau_{xz} = \tau_{zx} = y^2z$, $\tau_{yz} = \tau_{zy} = x^3y$.

Solution Find component of Body force at (1, 2, 3)

Equilibrium eq-① $\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\tau_{xy} + \frac{\partial}{\partial z}\tau_{xz} + \gamma_x = 0$

$$\frac{\partial}{\partial x}(20x^2 + y^2) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(y^2z) + \gamma_x = 0$$

$$\Rightarrow 40x + 0 + y^2 + \gamma_x = 0 \Rightarrow \gamma_x = -(40x + y^2) = -(40 \times 1 + 2^2) = \underline{-44}$$

Equilibrium eq-② $\frac{\partial}{\partial x}\tau_{yz} + \frac{\partial}{\partial y}\sigma_y + \frac{\partial}{\partial z}\tau_{yz} + \gamma_y = 0$

$$\Rightarrow \frac{\partial}{\partial z}(zx) + \frac{\partial}{\partial y}(30z^3 + 200) + \frac{\partial}{\partial z}(x^3y) + \gamma_y = 0 \Rightarrow z + 0 + 0 + \gamma_y = 0$$

$$\Rightarrow \boxed{\gamma_y = -3}$$

Equilibrium eq-③ $\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{zy} + \frac{\partial}{\partial z}\sigma_z + \gamma_z = 0$

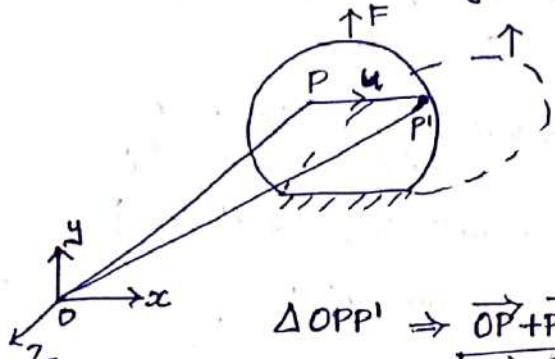
$$\Rightarrow \frac{\partial}{\partial x}(y^2z) + \frac{\partial}{\partial y}(x^3y) + \frac{\partial}{\partial z}(30(y^2 + z^2)) + \gamma_z = 0$$

$$\Rightarrow 0 + x^3 + 60z + \gamma_z = 0 \Rightarrow \gamma_z = -1^3 - 60 \times 3 = \underline{-181}$$

Strain

Displacement Field

- Elastic solid under equilibrium deform under action of force
- Deformation quantitatively expressed - Displacement of point of body



- Elastic body subjected to force F
- Solid curve - original configuration
- Dotted curve - Deformed Configuration
- P - Point of original body
- P' - point of deformed body

$$\Delta OPP' \Rightarrow \overrightarrow{OP} + \overrightarrow{PP'} = \overrightarrow{OP'}$$

$$[\overrightarrow{PP'} = \overrightarrow{OP'} - \overrightarrow{OP}]$$

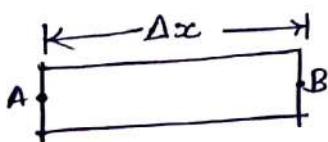
- Here PP' called displacement at a point P . Denoted by u [x direction]
- Similarly v, w are displacement in y and z direction

$$U(x, y, z) = u(x, y, z)\hat{i} + v(x, y, z)\hat{j} + w(x, y, z)\hat{k}$$

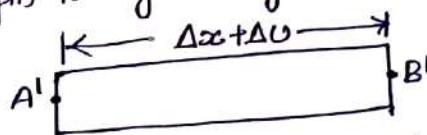
Above expression shows displacement is vector point function. Hence it is called displacement Field

Engineering Strain

Strain \rightarrow Ratio of change in length to original length



Original body with linear element strip AB



Deformation of element strip along x-direction

Normal Strain

Normal Strain (AB) =

$$\frac{\text{change in length}}{\text{Original length}} = \frac{\Delta x + \Delta u - \Delta x}{\Delta x} = \frac{\Delta u}{\Delta x}$$

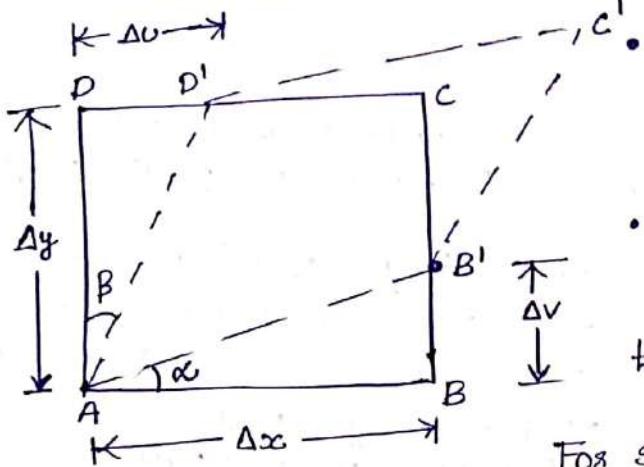
$$\epsilon_{xx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$$

Normal strain in x direction

$$\epsilon_y = \frac{\partial v}{\partial y} \rightarrow \text{Normal strain in y direction}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \rightarrow \text{Normal strain in z direction}$$

Shear Strain ($\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$)



The angle BAD is originally right angle changes to B'AD' as shown in figure.

- The Shear deformation is called Shear strain

$$\tan \alpha = \frac{\Delta v}{\Delta x} \quad \tan \beta = \frac{\Delta u}{\Delta y}$$

For smaller angle of α and β

$$\alpha = \frac{dv}{dx}, \beta = \frac{du}{dy} \Rightarrow \text{Total deformation } \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}$$

Generalise as

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$	Shear strain x-y plane
$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$	Shear strain y-z plane
$\gamma_{xz} = \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}$	Shear strain x-z plane

Strain Tensor: consider small deformation

Small Deformation \rightarrow Partial derivative of component of displacement are small.

$$\text{Strain tensor } [\epsilon] = \frac{1}{2} (\nabla u + \nabla u^T)$$

$$\text{Matrix of strain tensor } [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

In Elastic problem the shear strain terms are expressed half of original shear strain

$$\epsilon_{xy} = \gamma_{xy}/2, \epsilon_{xz} = \gamma_{xz}/2, \epsilon_{yz} = \gamma_{yz}/2$$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_{yy} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\ \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \\ \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

By heart Strain-displacement relation

$$\text{Normal strain} \rightarrow \epsilon_x = \frac{\partial u}{\partial x}, \epsilon_y = \frac{\partial v}{\partial y}, \epsilon_z = \frac{\partial w}{\partial z}$$

$$\text{Shear strain} \rightarrow \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \gamma_{xz} = \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}$$

① Displacement field of a body given by $\vec{u} = (x^2+y)\hat{i} + (3+z)\hat{j} + (x^2+2y)\hat{k}$
 Estimate for point (1, 2, 3)

- (a) component of Engineering strain
 (b) Arranged as a tensorial strain

Solution

(a) where $u = x^2+y$, $v = 3+z$, $w = x^2+2y$

Normal strain $\left\{ \begin{array}{l} \epsilon_x = \frac{\partial u}{\partial x} = 2x \Rightarrow \text{at } (1, 2, 3) = 2 \times 1 = \underline{\underline{2}} \\ \epsilon_y = \frac{\partial v}{\partial y} = 0, \quad \epsilon_z = \frac{\partial w}{\partial z} = 0 \end{array} \right.$

Shear strain

$$\frac{\gamma_{xy}}{2} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} (1+0) = \underline{\underline{\frac{1}{2}}}$$

$$\frac{\gamma_{yz}}{2} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} (1+2) = \underline{\underline{\frac{3}{2}}}$$

Tensorial strain

$$\text{② } [\epsilon] = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ 1 & \frac{3}{2} & 0 \end{bmatrix} \quad \frac{\gamma_{xz}}{2} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} (0+2x) = \underline{\underline{\frac{2}{2}}} = \underline{\underline{1}}$$

H.W ② A displacement field given by $\vec{u} = (y^2)\hat{i} + 3yz\hat{j} + (4+6x^2)\hat{k} \times 10^{-2}$
 what are the strain components at (1, 0, 2)

Solution

$$u = y^2 \times 10^{-2}, \quad v = 3yz \times 10^{-2}, \quad w = (4+6x^2) \times 10^{-2}$$

$$\epsilon_x = \frac{\partial u}{\partial x} = 0, \quad \epsilon_y = \frac{\partial v}{\partial y} = 3z \times 10^{-2} \Rightarrow \text{at } (1, 0, 2) = \underline{\underline{6 \times 10^{-2}}}, \quad \epsilon_z = \frac{\partial w}{\partial z} = 0$$

$$\frac{\gamma_{xy}}{2} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} (0+0) \times 10^{-2} \Rightarrow \text{at } (1, 0, 2) = \underline{\underline{0}}$$

$$\frac{\gamma_{yz}}{2} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} (3y+0) \times 10^{-2} \Rightarrow \text{at } (1, 0, 2) = \underline{\underline{3 \times 0}} = \underline{\underline{0}}$$

$$\frac{\gamma_{xz}}{2} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} (0+12z) \times 10^{-2} \Rightarrow \text{at } (1, 0, 2) = \underline{\underline{12 \times 1}} = \underline{\underline{6 \times 10^{-2}}}$$

$$\text{Strain Matrix } (\epsilon) = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 6 & 0 \\ 6 & 0 & 0 \end{bmatrix} \times 10^{-2} = \underline{\underline{}}$$

- ③ The components of displacement in a parallelopiped as shown in fig. $\vec{u} = axyz$, $v = bxzy$, $w = cxyz$ along $x, y \& z$ direction. The point F changes to $(1.504, 1.002, 1.996)$ using this data.

(a) unknown coefficient of displacement field

(b) obtain strain tensor at E

(c) Find normal strain along solid diagonal EA

Solution

$$\text{Co-ordinates } F \rightarrow (1.5, 1, 2)$$

$$E \rightarrow (1.5, 1, 0)$$

$$A \rightarrow (0, 0, 2)$$

(a) displacement = Deformed point - Original point

$$u = axyz$$

$$1.504 - 1.5 = a(1.5 \times 1 \times 2) \Rightarrow a = 0.0013$$

$$v = bxzy$$

$$1.002 - 1 = b(1.5 \times 1 \times 2) \Rightarrow b = 6.67 \times 10^{-4}$$

$$w = cxyz$$

$$1.996 - 2 = c(1.5 \times 1 \times 2)$$

$$c = -0.0013$$

b)

$$\left. \begin{array}{l} \epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(axyz) = ayz = 0.0013 \times 1 \times 0 = 0 \\ \epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(bxzy) = bxz = 6.67 \times 10^{-4} \times 1.5 \times 0 = 0 \\ \epsilon_z = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(cxyz) = cxy = -0.0013 \times 1.5 \times 1 = -1.95 \times 10^{-3} \\ \gamma_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} (axz + byz) = 0 \\ \gamma_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} (bxz + cxz) = \frac{1}{2} (6.67 \times 10^{-4} \times 1.5 \times 1) = 5.025 \times 10^{-4} \\ \gamma_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} (axy + cyz) = \frac{1}{2} (0.0013 \times 1.5 \times 1) = +9.75 \times 10^{-4} \end{array} \right\}$$

$$[e] = \begin{bmatrix} 0 & 0 & 9.75 \times 10^{-4} \\ 0 & 0 & 5.025 \times 10^{-4} \\ 9.75 \times 10^{-4} & 5.025 & -1.95 \times 10^{-3} \end{bmatrix}$$

c) Normal strain along two point

$$[e] = [n_x \ n_y \ n_z] [e_{ij}] \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$[e] = \begin{bmatrix} 0.557 & -0.38 & 0.75 \end{bmatrix} \begin{bmatrix} 0 & 0 & 9.75 \times 10^{-4} \\ 0 & 0 & 5.025 \times 10^{-4} \\ 9.75 \times 10^{-4} & 5.025 & -1.95 \times 10^{-3} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$[e] = -2.4 \times 10^{-4}$$

$$A \rightarrow (0, 0, 2)$$

$$B \rightarrow (1.5, 1, 0)$$

$$\vec{EA} = (0-1.5)\hat{i} + (0-1)\hat{j} + (2-0)\hat{k}$$

$$= -1.5\hat{i} - \hat{j} + 2\hat{k}$$

$$n_x = \frac{-1.5}{\sqrt{(-1.5)^2 + (-1)^2 + 2^2}} = -0.557$$

$$n_y = \frac{-1}{\sqrt{(-1.5)^2 + (-1)^2 + 2^2}} = -0.38$$

$$n_z = \frac{2}{\sqrt{(-1.5)^2 + (-1)^2 + 2^2}} = 0.75$$

Principal Strain & principal direction

Extreme value of normal strain at a point is called principal strain & corresponding direction are principal direction

$$\text{Strain tensor } [\epsilon] = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{bmatrix}$$

- First strain invariant $I_1 = \epsilon_x + \epsilon_y + \epsilon_z$

- Second strain invariant $I_2 = \left| \begin{array}{cc} \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zy}}{2} & \epsilon_z \end{array} \right| + \left| \begin{array}{cc} \epsilon_x & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{zx}}{2} & \epsilon_z \end{array} \right| + \left| \begin{array}{cc} \epsilon_x & \frac{\gamma_{xy}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y \end{array} \right|$

- Third strain invariant $I_3 = \left| \begin{array}{ccc} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{array} \right|$

Characteristic eq. is

$$[\epsilon^3 - I_1\epsilon^2 + I_2\epsilon - I_3 = 0]$$

The root of $\epsilon_1, \epsilon_2, \epsilon_3$ are called principal strain

① Tensorial strain is

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(a) Find 3 strain invariants

(b) write characteristic polynomial

(c) Extreme value of principal strain

(d) Check for invariance

(a) $I_1 = \epsilon_x + \epsilon_y + \epsilon_z = 3 + 5 + 3 = \underline{\underline{11}}$

$$I_2 = \left| \begin{array}{cc} 5 & -1 \\ -1 & 3 \end{array} \right| + \left| \begin{array}{cc} 3 & 1 \\ 1 & 3 \end{array} \right| + \left| \begin{array}{cc} 3 & -1 \\ -1 & 5 \end{array} \right| = \underline{\underline{36}}$$

$$I_3 = \left| \begin{array}{ccc} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{array} \right| = \underline{\underline{36}}$$

(b) $\epsilon^3 - I_1\epsilon^2 + I_2\epsilon - I_3 = 0 \Rightarrow [\epsilon^3 - 11\epsilon^2 + 36\epsilon - 36 = 0]$ (c) $\epsilon_1 = 6, \epsilon_2 = 3, \epsilon_3 = 2$

(d) $I_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 = \underline{\underline{11}}, I_2 = \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_1\epsilon_3 = 6 \times 3 + 3 \times 2 + 2 \times 6 = \underline{\underline{36}}$
 $I_3 = \epsilon_1 \epsilon_2 \epsilon_3 = 6 \times 3 \times 2 = 36$

② The displacement field of body is given by $u = [(x^2+y^2)\hat{i} + (y+10)\hat{j} + (x^2-2z^2)\hat{k}] \times 10^{-3}$

Determine (a) strain tensor at (2, 2, 3)

(b) Find 3 strain invariants

(c) write characteristic polynomial

(d) Extreme value of strain

(e) Determine direction of max principal strain

$$(a) \quad u = x^2 + y, \quad v = y + 10, \quad w = (x^2 - 2z^2)$$

$$\text{at } (2, 2, 3) \quad \left\{ \begin{array}{l} \epsilon_x = \frac{\partial u}{\partial x} = 2x = 2 \times 2 = 4, \quad \epsilon_y = \frac{\partial v}{\partial y} = 1, \quad \epsilon_z = \frac{\partial w}{\partial z} = -4z = -4 \times 3 = -12 \\ \gamma_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{-1}{2} = \frac{1}{2} = \underline{\underline{0}} \quad \gamma_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \gamma_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} (0+0) = \underline{\underline{0}} \quad = \frac{1}{2} (0+2x) = \underline{\underline{2}} \end{array} \right.$$

$$[\epsilon] = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

$$(b) \quad J_1 = \epsilon_x + \epsilon_y + \epsilon_z = 1 + 1 - 12 = \underline{\underline{-10}}$$

$$J_2 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -12 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 0 & -12 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \underline{\underline{-64}}$$

$$J_3 = \begin{vmatrix} 4 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{vmatrix} = \underline{\underline{-4}}$$

$$(c) \quad \boxed{\epsilon^3 - I_1 \epsilon^2 + I_2 \epsilon - I_3 = 0} \Rightarrow \boxed{\epsilon^3 + 7\epsilon^2 - 60 \cdot 25\epsilon + 49 = 0}$$

$$(d) \quad \epsilon_1 = \underline{\underline{5.18 \times 10^{-3}}}, \quad \epsilon_2 = \underline{\underline{0.06 \times 10^{-3}}}, \quad \epsilon_3 = \underline{\underline{-12.24 \times 10^{-3}}}$$

$$(e) \quad \begin{bmatrix} 4-4.82 & 2 & 2 \\ 2 & 1-4.82 & 0 \\ 2 & 0 & -12-4.82 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -0.32n_x + 2n_y + 2n_z = 0 \quad \text{--- (1)} \\ 2n_x - 3.32n_y + 0n_z = 0 \quad \text{--- (2)} \\ 2n_x + 0n_y - 16.32n_z = 0 \quad \text{--- (3)} \end{array}$$

$$\text{eq-(3)} \quad 2n_x = 16.32n_z \Rightarrow n_x = \frac{8.16n_z}{3.32} \Rightarrow n_z = \frac{1}{8.16} n_x \Rightarrow n_z = 0.123n_x$$

$$\text{eq-(2)} \quad 2n_x = 3.32n_y \Rightarrow n_y = \frac{2}{3.32} n_x \Rightarrow n_y = 0.608n_x$$

$$\boxed{n_x^2 + n_y^2 + n_z^2 = 1} \Rightarrow n_x^2 + (0.608n_x)^2 + (0.123n_x)^2 = 1 \\ \Rightarrow n_x^2 (1 + 0.362 + 0.015) = 1 \Rightarrow \underline{\underline{n_x^2 = 0.726}} n_x = \underline{\underline{0.852}}$$

$$n_y = \underline{\underline{0.573}}, \quad n_z = \underline{\underline{0.104796}}$$

Analogy Between Stress and Strain Tensor

Stress Tensor	Strain Tensor
* $[\sigma]_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$	* $[\epsilon]_{ij} = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{bmatrix}$
* Stress invariance are	* Strain invariance
$J_1 = \sigma_x + \sigma_y + \sigma_z$	$J_1 = \epsilon_x + \epsilon_y + \epsilon_z$
$J_2 = \left \begin{matrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{matrix} \right + \left \begin{matrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{matrix} \right + \left \begin{matrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{matrix} \right $	$J_2 = \left \begin{matrix} \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zy}}{2} & \epsilon_z \end{matrix} \right + \left \begin{matrix} \epsilon_x & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{zx}}{2} & \epsilon_z \end{matrix} \right + \left \begin{matrix} \epsilon_x & \frac{\gamma_{xy}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y \end{matrix} \right $
$J_3 = \sigma_{ij} $	$J_3 = \epsilon_{ij} $
* Extreme value of principal stresses are $\sigma_1, \sigma_2, \sigma_3$	* Extreme value of principal strains are $\epsilon_1, \epsilon_2, \epsilon_3$
* Hydrostatic stress tensor	* Hydrostatic strain tensor
$P = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$	$\begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix}$
$\boxed{P = \frac{\sigma_x + \sigma_y + \sigma_z}{3}}$	$\boxed{P = \frac{\epsilon_x + \epsilon_y + \epsilon_z}{3}}$
* Stress tensor is third order	* Strain tensor is third order
* Transformation rule applicable for both stress & strain tensor	
* Both stress & strain tensor	symmetric in nature
* Value of shear stress having coefficient = 1	* Value of shear strain having coefficient $= \frac{1}{2}$

(Q.III) Compatibility Conditions :

- When component of displacement u, v and w are available directly we can determine strain component easily by differentiating with respect to x, y & z . But when it is required to determine displacement from given strain. It is quite complex process. To integrate differential relationship.

In order to determine displacement, there exist certain relationship b/w normal & shear strain. That relations are called Compatibility Conditions

SET-①

$$\epsilon_x = \frac{\partial u}{\partial x} \quad (a) \qquad \epsilon_y = \frac{\partial v}{\partial y} \quad (b)$$

[Twice] Differentiate eqn (a) w.r.t. y and (b) w.r.t. x

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2}{\partial y^2} \left(\frac{\partial u}{\partial x} \right) \rightarrow \frac{\partial^2}{\partial y^2} (\epsilon_x) = \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial u}{\partial y} \right) \quad (1)$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left(\frac{\partial v}{\partial y} \right) \rightarrow \frac{\partial^2}{\partial x^2} (\epsilon_y) = \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial v}{\partial x} \right) \quad (2)$$

Adding L.H.S & R.H.S of eq. 1

$$\frac{\partial^2}{\partial y^2} \epsilon_x + \frac{\partial^2}{\partial z^2} \epsilon_y = \frac{\partial^2}{\partial x^2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} \right) = \frac{\partial^2}{\partial x^2} (\gamma_{xy}) \quad \text{--- ①}$$

Similarly

$$\frac{\partial^2}{\partial z^2} \epsilon_y + \frac{\partial^2}{\partial x^2} \epsilon_z = \frac{\partial^2}{\partial y^2} (\gamma_{yz}) \quad \text{--- ②}$$

$$\frac{\partial^2}{\partial x^2} \epsilon_x + \frac{\partial^2}{\partial z^2} \epsilon_z = \frac{\partial^2}{\partial x^2} (\gamma_{xz}) \quad \text{--- ③}$$

→ SET-①

SET-②

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}} \Rightarrow \text{Differentiate w.r.t. } z$$

$$\frac{\partial}{\partial z} \gamma_{xy} = \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} \right) = \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 v}{\partial z^2} \quad \text{--- ④}$$

also we have

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \Rightarrow \text{differentiate w.r.t. } x$$

$$\frac{\partial}{\partial x} \gamma_{yz} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y} \quad \text{--- ⑤}$$

Similarly $\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \Rightarrow \text{differentiating w.r.t. } y$

$$\frac{\partial}{\partial y} \gamma_{xz} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial y \partial x} \quad \text{--- ⑥}$$

$$\Rightarrow ④ + ⑤ - ⑥ \Rightarrow \frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} - \frac{\partial}{\partial y} \gamma_{xz} = \cancel{\frac{\partial^2 u}{\partial z \partial y}} + \cancel{\frac{\partial^2 v}{\partial z \partial x}} + \cancel{\frac{\partial^2 w}{\partial z \partial y}} - \cancel{\frac{\partial^2 u}{\partial z \partial z}} - \cancel{\frac{\partial^2 v}{\partial z \partial y}} - \cancel{\frac{\partial^2 w}{\partial z \partial x}}$$

$$\Rightarrow \frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} - \frac{\partial}{\partial y} \gamma_{xz} = 2 \frac{\partial^2 v}{\partial x \partial z}$$

Differentiating w.r.t. y

$$\frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} - \frac{\partial}{\partial y} \gamma_{xz} \right) = 2 \frac{\partial}{\partial y} \left(\frac{\partial^2 v}{\partial x \partial z} \right) = 2 \frac{\partial^2}{\partial y \partial z} \epsilon_x \quad \text{--- ⑦}$$

Similarly

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \gamma_{xz} + \frac{\partial}{\partial z} \gamma_{xy} - \frac{\partial}{\partial x} \gamma_{yz} \right) = 2 \frac{\partial^2}{\partial y \partial z} \epsilon_x \quad \text{--- ⑧}$$

$$\frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} \gamma_{yz} + \frac{\partial}{\partial y} \gamma_{xz} - \frac{\partial}{\partial z} \gamma_{xy} \right) = 2 \frac{\partial^2}{\partial y \partial z} \epsilon_z \quad \text{--- ⑨}$$

① Determine whether the following strain field compatible or not

$$\epsilon_x = 2x^2 + 3y^2 + z + 1 \quad \epsilon_y = x^2 + 2y^2 + 3xz + 2 \quad \epsilon_z = 3x + 2y + 1 + z^2$$

$$\gamma_{xy} = 8xy \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$$

Solution

$$\text{eq-1} \quad \frac{\partial^2}{\partial y^2} \epsilon_x + \frac{\partial^2}{\partial x^2} \epsilon_y = \frac{\partial^2}{\partial x \partial y} \gamma_{xy} \quad \text{--- (1)}$$

$$6 + 2 = 8 \quad \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

$$\text{eq-2} \quad \frac{\partial^2}{\partial z^2} \epsilon_y + \frac{\partial^2}{\partial y^2} \epsilon_z = \frac{\partial^2}{\partial y \partial z} \gamma_{yz} \quad \text{--- (2)}$$

$$0 + 0 = 0 \quad \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

$$\text{eq-3} \quad \frac{\partial^2}{\partial x^2} \epsilon_z + \frac{\partial^2}{\partial z^2} \epsilon_x = \frac{\partial^2}{\partial z \partial x} \gamma_{zx} \quad \text{--- (3)}$$

$$0 + 0 = 0 \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

$$\text{Similarly } \frac{\partial}{\partial y} \left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} \right] = \frac{\partial^2 \epsilon_y}{\partial z \partial x} \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} [0+0+0] = 2 \frac{\partial \epsilon_y}{\partial z} \Rightarrow 0=0 \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Similarly ~~(5)~~ you must prove eq (5) & (6)

Hence proved

② For a plane strain case, the strains are specified as, under state whether they are compatible or not. Justify your answer

$$\epsilon_x = 3x^2y \quad \epsilon_y = 4y^2x \quad \gamma_{xy} = x^2 + xy$$

Solution

$$\text{eq-1} \quad \frac{\partial^2}{\partial y^2} \epsilon_x + \frac{\partial^2}{\partial x^2} \epsilon_y = \frac{\partial^2}{\partial x \partial y} \gamma_{xy} \quad \text{--- (1)}$$

$$0+0 \neq 1 \Rightarrow \text{L.H.S.} \neq \text{R.H.S.}$$

It is not compatible

[Here z component zero \Rightarrow no check out]

eq-1

$$\text{where } \Rightarrow \frac{\partial}{\partial y} \epsilon_x = 6y \Rightarrow \frac{\partial^2 \epsilon_x}{\partial y^2} = 6$$

$$\frac{\partial}{\partial x} \epsilon_y = 2x \Rightarrow \frac{\partial^2 \epsilon_y}{\partial x^2} = 2$$

$$\frac{\partial}{\partial x} (\gamma_{xy}) = 8y \Rightarrow \frac{\partial^2 (\gamma_{xy})}{\partial x \partial y} = 8$$

$$\frac{\partial}{\partial z} \epsilon_y = 3, \quad \frac{\partial^2}{\partial z^2} = 0$$

$$\frac{\partial}{\partial y} \epsilon_z = 2, \quad \frac{\partial^2}{\partial y^2} \epsilon_z = 0$$

$$\frac{\partial}{\partial y} \gamma_{yz} = 0, \quad \frac{\partial^2}{\partial y \partial z} \gamma_{yz} = 0$$

$$\frac{\partial}{\partial x} \epsilon_z = 3, \quad \frac{\partial^2}{\partial x^2} \epsilon_z = 0$$

$$\frac{\partial}{\partial z} \epsilon_x = 1, \quad \frac{\partial^2}{\partial z^2} \epsilon_x = 0$$

$$\frac{\partial}{\partial z} \gamma_{zx} = 0, \quad \frac{\partial^2}{\partial z \partial x} \gamma_{zx} = 0$$

$$\frac{\partial}{\partial y} \epsilon_x = 3x^2, \quad \frac{\partial^2}{\partial y^2} \epsilon_x = 0$$

$$\frac{\partial}{\partial x} \epsilon_y = 4y^2, \quad \frac{\partial^2}{\partial x^2} \epsilon_y = 0$$

$$\frac{\partial}{\partial x} (\gamma_{xy}) = 2x + y$$

$$\frac{\partial^2}{\partial x \partial y} (\gamma_{xy}) = 1$$