AMOS Module-3

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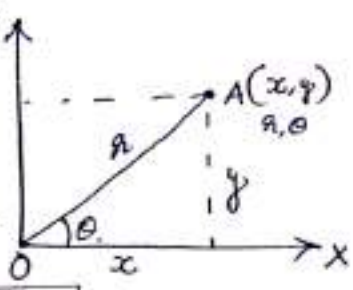
Equations in polar Co-ordinate System (2D)

polar Co-ordinate System

- Consider A having Co-ordinate (x,y) in Rectangular System at distance x and y along x-y direction

  θ = Angular movement

  r = Linear distance radial direction



In 2D Cartesian (x,y) System
polar (r,θ)

$$x = r\cos\theta \qquad x^2+y^2 = r^2$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Equation in polar Coordinate System

unit Vector in Cartesian : $\hat{i}$ and $\hat{j}$

unit Vector in Polar : e_r and e_θ

Relation b/w e_r, e_θ & $\hat{i}$ and $\hat{j}$

$$e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$e_z = \hat{k}$$

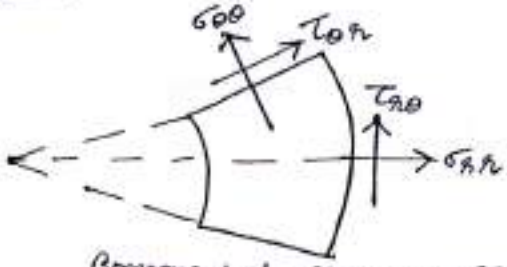
Matrix Form

$$\begin{bmatrix} \hat{e_r} \\ \hat{e_\theta} \\ e_z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=[Q]\text{ - Transformation Matrix}} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Stress Tensor in 2D polar Coordinate System

Stress tensor :

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & 0 \\ \tau_{\theta r} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$



Component of Stress at a point in polar Co-ordinate System

Relation b/w Cartesian & polar Stress tensor

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & 0 \\ \tau_{\theta r} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = [Q] \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} [Q]^T$$

where $[Q]$ = Transformation Matrix

Note: $\sigma_{rr}+\sigma_{\theta\theta} = \sigma_{xx}+\sigma_{yy}$

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Strain in 2D polar Co-ordinate System

Displacement Field in 2D polar Co-ordinate System

$u = u_r e_r + u_\theta e_\theta + u_z e_z$ (Similarly) $u = u\hat{i} + v\hat{j} + w\hat{k}$

where u_r, u_θ displacement along r, θ, z direction

Strain - displacement Relation (2D polar)

Component of Eg. Strain in 2D polar co-ordinate system $\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}$ and $\gamma_{r\theta}$

Here $\epsilon_{rr} \to \epsilon_{xx}$, $\epsilon_{\theta\theta} \to \epsilon_{yy}$, $\gamma_{r\theta} \to \gamma_{xy}$

r - Radial direction

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

Imp

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

Strain tensor (2D polar)

$$\begin{bmatrix} \epsilon_{rr} & \frac{\gamma_{r\theta}}{2} & 0 \\ \frac{\gamma_{\theta r}}{2} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & 0 \\ \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) & 0 \\ 0 & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Equilibrium Equations polar Co-ordinate System (2D)

- Equilibrium eq: (2D) polar coordin for balance force Radial & tangential direction
- Hence equilibrium eq: are called radial equilibrium eq. and tangential equilibrium eq:

The Equations of Equilibrium of 2D polar Coordinate System

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\sigma_{rr})\right] + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\sigma_{\theta\theta}}{r} + b_r = 0 \quad \text{(i)}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\tau_{r\theta})\right] + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{r} + b_\theta = 0 \quad \text{(ii)}$$

where b_r and b_θ are components of body force term along radial & tangential direction

Above equation resemblance with equilibrium eq: of Cartesian Co-ordinate (2D)

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{xy} + b_x = 0$$

$$\frac{\partial}{\partial x}\tau_{yx} + \frac{\partial}{\partial y}\sigma_{yy} + b_y = 0$$

Here,
- $\frac{\partial}{\partial x}$ replace with $\frac{1}{r}\left[\frac{\partial}{\partial r}r\right]$ & σ_{xx} with σ_{rr}
- $\frac{\partial}{\partial y}$ replace with $\frac{1}{r}\frac{\partial}{\partial \theta}$ & τ_{xy} with $\tau_{r\theta}$
- one Additional term $\frac{\sigma_{\theta\theta}}{r}$ in the radial force
- one Additional term $\frac{\tau_{r\theta}}{r}$ in the tangential force

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Airy's Stress Function in polar Co-ordinates

Relation connecting cartesian & polar coordinate system

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \qquad \frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y \qquad \frac{\partial y}{\partial \theta} = r\cos\theta = x$$

ϕ = Airy's Stress Function (Scalar)

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}$$

We know that

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & 0 \\ \tau_{\theta r} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx}\cos\theta + \tau_{yx}\sin\theta & \tau_{xy}\cos\theta + \sigma_{yy}\sin\theta & 0 \\ -\sigma_{xx}\sin\theta + \tau_{yx}\cos\theta & -\tau_{xy}\sin\theta + \sigma_{yy}\cos\theta & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & 0 \\ \tau_{\theta r} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx}\cos^2\theta + \tau_{yx}\sin\theta\cos\theta + \tau_{xy}\cos\theta\sin\theta + \sigma_{yy}\sin^2\theta & -\sigma_{xx}\cos\theta\sin\theta - \tau_{yx}\sin^2\theta + \tau_{xy}\cos^2\theta + \sigma_{yy}\sin\theta\cos\theta & 0 \\ -\sigma_{xx}\sin\theta\cos\theta + \tau_{yx}\cos^2\theta - \tau_{xy}\sin^2\theta + \sigma_{yy}\sin\theta\cos\theta & \sigma_{xx}\sin^2\theta - \tau_{yx}\sin\theta\cos\theta - \tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos^2\theta & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

Where $\sigma_{rr} = \cos^2\theta\,\sigma_{xx} + 2\tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$

$$= \cos^2\theta\frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\,\partial y}\sin\theta\cos\theta + \sin^2\theta\frac{\partial^2\phi}{\partial x^2}$$

$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta - 2\tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos^2\theta$

$$= \sin^2\theta\frac{\partial^2\phi}{\partial y^2} + 2\frac{\partial^2\phi}{\partial x\,\partial y}\sin\theta\cos\theta + \cos^2\theta\frac{\partial^2\phi}{\partial x^2}$$

$\tau_{r\theta} = \sin\theta\cos\theta(\sigma_{yy} - \sigma_{xx}) + (\cos^2\theta - \sin^2\theta)\tau_{xy}$

$$= \sin\theta\cos\theta\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right) - (\cos^2\theta - \sin^2\theta)\frac{\partial^2\phi}{\partial x\,\partial y}$$

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Conversion of Airy's Equation

(Derive Stress Component in polar co-ordinate in terms of Airy's stress function)

Derive Bi-harmonic eq: in polar co-ordinate

Components of polar co-ordinate System are defined interms as derivative of Airy's stress function.

It done by converting corresponding expression of Cartesian Co-ordinate System.

Derivation of polar normal Stress components in terms of ϕ

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial \phi}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial \phi}{\partial y}\left(\frac{y}{r}\right)$$

$$\boxed{r\frac{\partial \phi}{\partial r} = x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}} \quad \text{—①}$$

operator

$$\boxed{r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}} \quad \text{—②}$$

① × ②

$$r\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}\right)$$

$$= x\frac{\partial}{\partial x}\left(x\frac{\partial \phi}{\partial x}\right) + x\frac{\partial}{\partial x}\left(y\frac{\partial \phi}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial \phi}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial \phi}{\partial y}\right)$$

$$= x^2\frac{\partial^2 \phi}{\partial x^2} + xy\frac{\partial^2 \phi}{\partial x \partial y} + xy\frac{\partial^2 \phi}{\partial x \partial y} + y^2\frac{\partial^2 \phi}{\partial y^2}$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{x^2}{r^2}\frac{\partial^2 \phi}{\partial x^2} + \frac{2xy}{r^2}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{y^2}{r^2}\frac{\partial^2 \phi}{\partial y^2}} \quad \text{—④}$$

Similarly

$$\frac{\partial \phi}{\partial \theta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial \phi}{\partial x}(-y) + \frac{\partial \phi}{\partial y}(x)$$

$$\frac{\partial \phi}{\partial \theta} = x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \quad \text{—③} \checkmark$$

operator $\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ —④

③ × ④

$$\frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial \theta}\right) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right)$$

$$= x^2\frac{\partial^2 \phi}{\partial y^2} - xy\frac{\partial^2 \phi}{\partial x \partial y} - xy\frac{\partial^2 \phi}{\partial x y} + y^2\frac{\partial^2 \phi}{\partial x^2}$$

$$= x^2\frac{\partial^2 \phi}{\partial y^2} - 2xy\frac{\partial^2 \phi}{\partial x \partial y} + y^2\frac{\partial^2 \phi}{\partial x^2}$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial \theta}\right) = \frac{x^2}{r^2}\frac{\partial^2 \phi}{\partial y^2} - \frac{2xy}{r^2}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{y^2}{r^2}\frac{\partial^2 \phi}{\partial x^2}} \quad \text{—⑤}$$

A + B ⇒

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial \theta}\right) = \frac{x^2}{r^2}\frac{\partial^2 \phi}{\partial x^2} + \frac{y^2}{r^2}\frac{\partial^2 \phi}{\partial x^2} + \frac{x^2}{r^2}\frac{\partial^2 \phi}{\partial y^2}$$

Imp

$\nabla^2 \phi = 0$ Biharmonic eq

$$\Rightarrow \underbrace{\frac{\partial^2 \phi}{\partial r^2}}_{=\sigma_{\theta\theta}} + \underbrace{\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{r^2 \partial \theta^2}}_{\sigma_{rr}} = \frac{\partial^2 \phi}{\partial x^2}\left(\frac{x^2+y^2}{r^2}\right) + \frac{\partial^2 \phi}{\partial y^2}\left(\frac{x^2+y^2}{r^2}\right)$$

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$\nabla^2 \phi$

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Plane Strain

Cartesian $\nabla^2(\sigma_{xx}+\sigma_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\right)$

polar $\nabla^2(\sigma_{rr}+\sigma_{\theta\theta}) = -\frac{1}{1-\nu}\left(\frac{\partial b_r}{\partial r}+\frac{b_r}{r}+\frac{\partial b_\theta}{r\partial\theta}\right)$

$(\nabla^2\phi)$

$$\boxed{\nabla^2(\nabla^2)\phi = -\frac{1}{1-\nu}\left(\frac{\partial b_r}{\partial r}+\frac{b_r}{r}+\frac{\partial b_\theta}{r\partial\theta}\right)}$$

① Show that $\phi(r,\theta) = \left(Ar^2+\frac{C}{r^2}+D\right)\cos 2\theta$ satisfy the Stress comp equation

(b) Find component of Stress

Solution

Given $\phi = \left(Ar^2+\frac{C}{r^2}+B\right)\cos 2\theta$

we must prove $\nabla^2\phi = \left(\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$

$$\nabla^2\phi = \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \quad \text{---(1)}$$

where $\frac{\partial\phi}{\partial r} = \left(2Ar-\frac{2C}{r^3}\right)\cos 2\theta$

$$\frac{\partial^2\phi}{\partial r^2} = \left(2A+\frac{6C}{r^4}\right)\cos 2\theta$$

$$\frac{\partial\phi}{\partial\theta} = \left(Ar^2+\frac{C}{r^2}+D\right)\times\sin 2\theta\times -2$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -4\cos 2\theta\left(Ar^2+\frac{C}{r^2}+D\right)$$

Eq. $\therefore \psi = 2A\cos 2\theta + \frac{6C\cos 2\theta}{r^4} + 2A - \frac{2C\cos 2\theta}{r^4} - 4\cos 2\theta Ar^2 - \frac{4\cos 2\theta C}{r^4} - \frac{4\cos 2\theta D}{r^2}$

$$\psi = \frac{-4D\cos 2\theta}{r^2}$$

Subeq-① ⇒ for substitution $\frac{\partial\psi}{\partial r} = +\frac{8D\cos 2\theta}{r^3}$

$$\frac{\partial^2\psi}{\partial r^2} = -\frac{24D\cos 2\theta}{r^4}, \quad \frac{\partial\psi}{\partial\theta} = +\frac{4D}{r^2}\sin 2\theta\times 2$$

Eq-① ⇒ $\nabla^4\phi = -\frac{24D\cos 2\theta}{r^4}+\frac{8D\cos 2\theta}{r^4}$ $\quad \frac{\partial^2\psi}{\partial\theta^2} = \frac{16D\cos 2\theta}{r^2}$

$+ 16D/r^4\cos 2\theta = 0$ //

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(b) $$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{r^2 \partial \theta^2} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \left(2A - \frac{6C}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r \partial \theta}\right) = \left(2A - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta$$

Stress-Strain Relations

(a) plane stress

Here $\sigma_{zz} = 0$, $\tau_{rz} = 0$, $\tau_{\theta z} = 0$ } zero

σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$ } non zero

$$\epsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta})$$

$$\epsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})$$

$$\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})$$

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G}$$

(b) plane strain

ϵ_{zz}, γ_{rz}, $\gamma_{z\theta}$ } zero

ϵ_{rr}, $\epsilon_{\theta\theta}$, $\gamma_{r\theta}$ } non zero

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ 0 \\ \gamma_{r\theta} \end{bmatrix}$$

where $\mu = \frac{E}{2(1+\nu)}$ and $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

Face area

(f) Sr.

Non

Area.

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(2) Show that function $\phi = Ar^4\cos 4\theta + Br^4\cos 2\theta$ qualifies as Stress function

Solution

Given $\phi = Ar^4\cos 4\theta + Br^4\cos 2\theta$

we must prove $\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$

ψ

$\Downarrow$ ①

$\nabla^2\phi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$ ①

where $\frac{\partial\phi}{\partial r} = 4Ar^3\cos 4\theta + 4Br^3\cos 2\theta$

$\frac{\partial^2\phi}{\partial r^2} = 12Ar^2\cos 4\theta + 12Br^2\cos 2\theta$

$\frac{\partial\phi}{\partial\theta} = -4Ar^4\sin 4\theta - 2Br^4\sin 2\theta$

$\frac{\partial^2\phi}{\partial\theta^2} = -16Ar^4\cos 4\theta - 4Br^4\cos 2\theta$

$\therefore \psi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 12Ar^2\cos 4\theta + 12Br^2\cos 2\theta$

$+ \frac{1}{r^2}\times 4Ar^3\cos 4\theta + \frac{1}{r^2}\times 4Br^3\cos 2\theta$

$+ \frac{1}{r^2}\times -16Ar^4\cos 4\theta - \frac{1}{r^2}\times 4Br^2\cos 2\theta$

$\cos 2\theta = 0$

$= 12Ar^2\cos 4\theta + 12Br^2\cos 2\theta + 4Ar^2\cos 4\theta + 4Br^2\cos 2\theta - 16Ar^2\cos 4\theta - 4Br^2\cos 2\theta$

$\psi = 12Br^2\cos 2\theta$

Sub: eq ① $\Rightarrow$ For substitution $\frac{\partial\psi}{\partial r} = 24Br\cos 2\theta$

$\frac{\partial^2\psi}{\partial r^2} = 24B\cos 2\theta$

$\frac{\partial\psi}{\partial\theta} = -12Br^2\sin 2\theta \times 2$

$\frac{\partial^2\psi}{\partial\theta^2} = -48Br^2\cos 2\theta$

Sub: eq ① $\nabla^4\phi = 24B\cos 2\theta + \frac{1}{r}\times 24Br\cos 2\theta + \frac{1}{r^2}\times -$

$= 0$ //

Hence function ϕ qualifies as Stress function

.0
.3
.20 x 2

D COOLE

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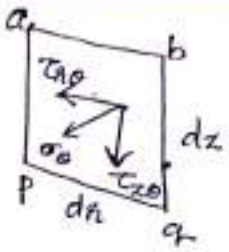
Equilibrium Equation in polar Co-ordinate System

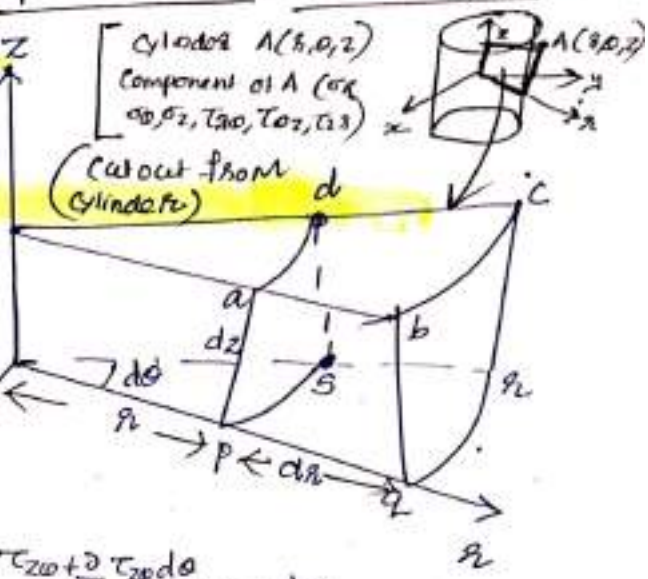
σ_R & σ_θ = Radial and circumferential components, σ_z = Axial component

[Cylinder A(R,θ,z) Component of A (σ_R, σ_θ, σ_z, $\tau_{R\theta}$, $\tau_{\theta z}$, τ_{zR})]

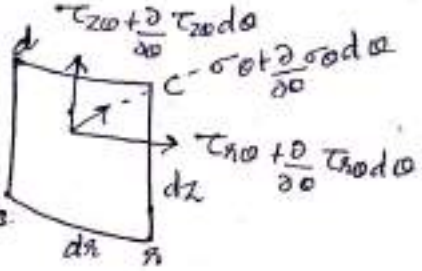
(a) Face apqb

- Area = $dR\,dz$
- Normal stress = σ_θ
- Shear stress = $\tau_{R\theta}$ & $\tau_{z\theta}$

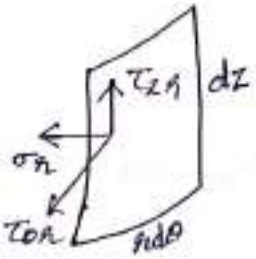




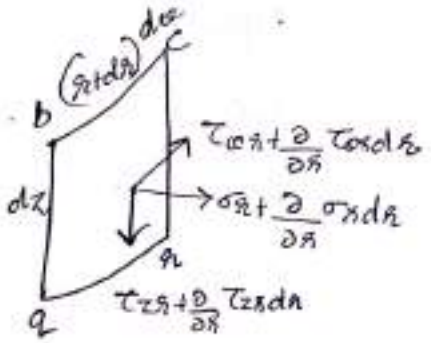
(b)
- Face dsRc
- Area = $dR\,dz$
- Normal stress = $\sigma_\theta + \frac{\partial}{\partial\theta}\sigma_\theta d\theta$
- Shear stresses = $\tau_{R\theta} + \frac{\partial}{\partial\theta}\tau_{R\theta}d\theta$ and $\tau_{z\theta} + \frac{\partial}{\partial\theta}\tau_{z\theta}d\theta$



(c)
- Face apsd
- Area = $R\,d\theta \times dz$
- Normal stress = σ_R
- Shear stresses = τ_{zR} and $\tau_{\theta R}$

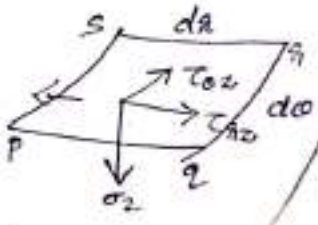


(d)
- Face bqRc
- Area = $(R+dR)\,d\theta\,dz$
- Normal stress = $\sigma_R + \frac{\partial}{\partial R}\sigma_R dz$
- Shear stresses = $\tau_{zR} + \frac{\partial}{\partial R}\tau_{zR}dR$ and $\tau_{\theta R} + \frac{\partial}{\partial R}\tau_{\theta R}dR$



Face pqRS

Area = $\left(R+\frac{dR}{2}\right)d\theta\,dR$



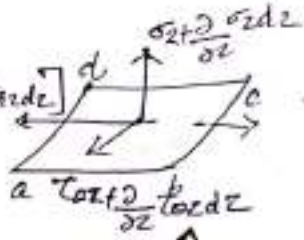
Face abcd

(f) Normal stress = σ_z & Shear stress τ_{Rz} & $\tau_{\theta z}$

Shear stress = $\tau_{Rz}+\frac{\partial}{\partial z}\tau_{Rz}dz$ & $\tau_{\theta z}+\frac{\partial}{\partial z}\tau_{\theta z}dz$

Normal stress = $\sigma_z+\frac{\partial}{\partial z}\sigma_z dz$

Area = $\left(R\,dR\,\frac{dR}{2}\right)d\theta\,dz$



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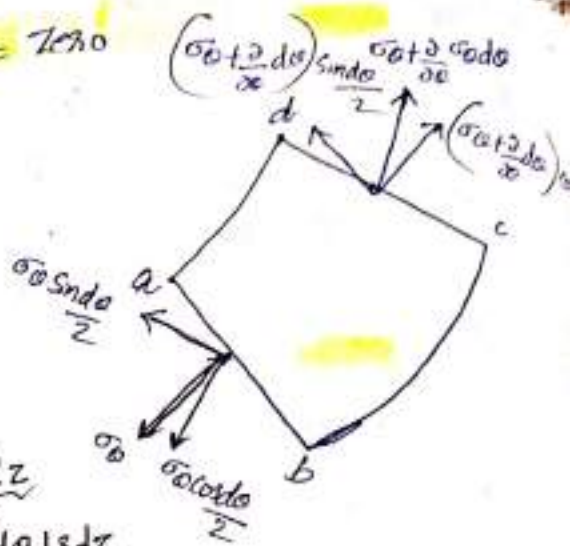
B_r, B_θ, B_z = Body Force component/unit volume

Equilibrium — Sum of Forces in r, θ, z direction = $\Sigma F_{r\theta} = 0$



Equating all forces in r direction

①

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r+dr)\,d\theta\,dz \quad ④$$
$$+ \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial z} dz\right)\left(r + \frac{dr}{2}\right) d\theta\,dr \quad ⑤ \quad - \sigma_r\, r\, d\theta\, dz \quad ⑥$$
$$- \tau_{rz}\left(r + \frac{dr}{2}\right) d\theta\, dr \quad ④ \quad - \sigma_\theta \sin\frac{d\theta}{2} dr\, dz \quad ⑤ \quad - \tau_{r\theta} \cos\frac{d\theta}{2} dr\, dz$$
$$- \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial r} dr\right) \sin\frac{d\theta}{2} dr\, dz \quad ③$$
$$+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta\right) \cos\frac{d\theta}{2} dr\, dz + B_r\left(r + \frac{dr}{2}\right) dr\, d\theta\, dz = 0 \quad ②$$

$\left(\sin\frac{d\theta}{2} \approx \frac{d\theta}{2} \;\&\; \cos\frac{d\theta}{2} = 1\right)$

expanding & cancelling / divided by $dr\,d\theta\,dz\,r$

$d\theta \to 0,\ dr \to 0,\ dz \to 0$

we get (3D) $\frac{\partial}{\partial r}\sigma_r + \frac{\partial}{\partial z}\tau_{rz} + \frac{1}{r}\frac{\tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0$

In (2D) $\boxed{\frac{\partial}{\partial r}\sigma_r + \frac{1}{r}\frac{\tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0}$

Equating all Forces in θ direction

$\sigma_\theta \times \underbrace{dr\,dz}_{\text{Area of } apqb} - \sigma_\theta + \frac{\partial}{\partial \theta}\sigma_\theta d\theta \times \underbrace{dr\,dz}_{\text{Area of } dsnc} + \tau_{\theta r} \times \underbrace{r\,d\theta\,dz}_{\text{Area of } apsd} - \tau_{\theta r} + \frac{\partial}{\partial r}\tau_{\theta r} dr \times \underbrace{(r+dr)}_{\text{Area of}}$

$+ \left(\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{\theta z} dz\right) \times \underbrace{(r + dr)\,d\theta\,dr}_{\text{Area of abcd}} - \tau_{\theta z} \times \underbrace{\left(r + \frac{dr}{2}\right) d\theta\, dz}_{\text{Area of pqrs}} + B_\theta$ (dsdθdz)

expanding & solving / divide by $dr\,d\theta\,dz$

$d\theta \to 0,\ dr \to 0,\ dz \to 0$

we get $\frac{\partial}{\partial r}\tau_{r\theta} + \frac{\partial}{\partial z}\tau_{\theta z} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} + B_\theta = 0$

In 2D $\boxed{\frac{\partial}{\partial r}\tau_{r\theta} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} = 0}$

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Axisymmetric Problems

- A solid formed by revolution of plane region about an axis (z-axis) called **axisymmetric solid**.

Plane strain / Plane stress

Deformation in these case also will be symmetrical about z-axis

∴ Stress component are independent of θ.

∴ $\tau_{r\theta} = \tau_{\theta z} = 0$

- Problem involving axisymmetric loading of an axisymmetric solid shell is called **axisymmetric problem**

Here Stress tensor in Axisymmetric problem $[\sigma_{ij}] = \begin{bmatrix} \sigma_{rr} & 0 & \tau_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \tau_{zr} & 0 & \sigma_{zz} \end{bmatrix}$

Strain-displacement Relation

Coordinates	x	y	z	$u = u\hat{i} + w\hat{k}$ (unit Vector)
Axisymm unit problem	r	θ	z	$u = u_r\hat{e}_r + u_z e_z$ (unit Vector)

Strain tensor $[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{rr} & 0 & \frac{\gamma_{rz}}{2} \\ 0 & \epsilon_{\theta\theta} & 0 \\ \frac{\gamma_{zr}}{2} & 0 & \epsilon_{zz} \end{bmatrix}$

$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta} = \frac{u_r}{r}$, $\epsilon_{zz} = \frac{\partial u_z}{\partial z}$

$\gamma_{rz} = \gamma_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$

- Stress - Strain Relations

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{rz} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{rz} \end{bmatrix}$$

where

$$\mu = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

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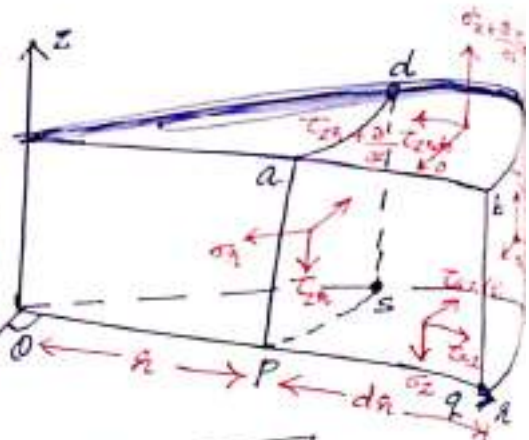
Equilibrium Equation in Axisymmetric Problem



- Area of face abcd = Area of face pqrs

$$= \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2} r^2 d\theta$$

$$= \frac{1}{2} r^2 d\theta + r\,dr\,d\theta + \frac{1}{2} dr^2 d\theta - \frac{1}{2} r^2 d\theta$$

$$\boxed{\text{Area of face} = \left(r + \frac{dr}{2}\right) dr\,d\theta}$$

$$= abcd = pqrs$$

- Stress are $\sigma_z + \frac{\partial}{\partial z}\sigma_z dz$ & $\tau_{rz} + \frac{\partial}{\partial z}\tau_{rz} dz$ (a+c+A)
- Stresses pqrs : σ_z and τ_{rz}
- Stresses apsd : σ_r and τ_{zr}
- Stresses bqrc : $\tau_{zr} + \frac{\partial}{\partial r}\tau_{zr} dr$ and $\sigma_r + \frac{\partial}{\partial r}\sigma_r dr$,

$$\text{Area of pqrs} = \left(r + \frac{dr}{2}\right) dr\,d\theta$$

$$\text{Area of apsd} = r\,d\theta\,dz$$

$$\text{Area of bqrc} = (r+dr)\,d\theta\,dz$$

For equilibrium in z direction, we can equate the sum of forces acting in z direction = zero

$$\left(\sigma_z + \frac{\partial}{\partial z}\sigma_z dz\right)\left(r+\frac{dr}{2}\right)d\theta\,dr + \left(\tau_{rz} + \frac{\partial}{\partial r}\tau_{rz} dr\right)(r+dr)\,d\theta\,dz$$

$$- \tau_{rz}\, r\,d\theta\,dz - \sigma_z \left(r+\frac{dr}{2}\right) d\theta\,dr + B_z \left(r+\frac{dr}{2}\right) d\theta\,dr\,dz = 0$$

(Area of apsd; Area of pqrs; Area of bqrc)

where B_z = Body Force / unit Volume in z-direction

on expanding & simplify we get

$$\frac{\partial \sigma_z}{\partial z}\left(r+\frac{dr}{2}\right) dr\,d\theta\,dz + \frac{\partial \tau_{rz}}{\partial r}(r+dr)\,dr\,d\theta\,dz + \tau_{rz}\,dr\,d\theta\,dz + B_z\left(r+\frac{dr}{2}\right) d\theta\,dr\,dz = 0$$

Dividing throughout by $r\,dr\,d\theta\,dz$ applying limit $dr \to 0$

$$\boxed{\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + B_z = 0} \quad \text{——— (1)}$$

Equating Forces in r direction = zero

$$\left(\tau_{zr} + \frac{\partial}{\partial z}\tau_{zr} dz\right) \times \left(r+\frac{dr}{2}\right) d\theta\,dz - \left(\sigma_r + \frac{\partial}{\partial r}\sigma_r dr\right)(r+dr)\,d\theta\,dz - \sigma_r \times r\,d\theta\,dz + \tau_{rz} \times (\text{sk}$$

(Area abcd; Area of bqrc; Area of apsd)

$$+ B_r \left(r+\frac{dr}{2}\right) d\theta\,dr\,dz = 0$$

expanding & solving $dr \to 0$

$$\boxed{\frac{\partial}{\partial r}\sigma_r + \frac{\partial}{\partial z}\tau_{rz} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0} \quad \text{——— (2)}$$

B_r = Body Force / unit Volume r-direction

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Stress in thick cylinders (problem of thick cylinders called lames problem)

- Stress function of thick cylinder $\phi(r) = A\log r + Cr^2 + D$
- Component of Stress

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr} = \frac{1}{r}\times\frac{A}{r}\times 1 + 2C\times r\times\frac{1}{r} = \frac{A}{r^2} + 2C \quad \text{—— (1)}$$

$$\sigma_{\theta\theta} = \frac{d^2\phi}{dr^2} = \frac{d}{dr}\left(\frac{d\phi}{dr}\right) = \frac{d}{dr}\left(\frac{A}{r} + 2Cr\right) = \frac{-A}{r^2} + 2C \quad \text{(2)}$$

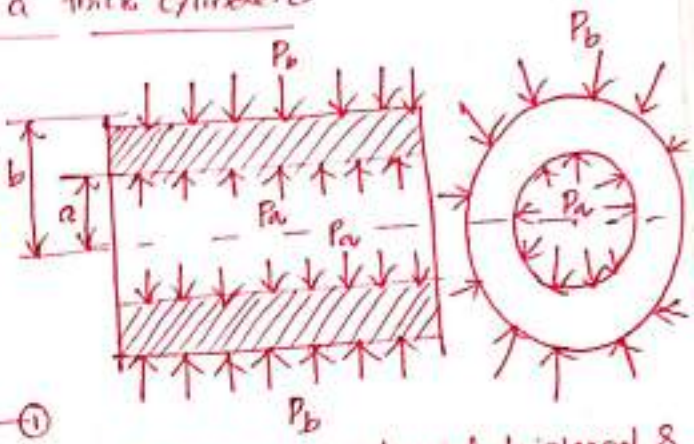
$$\tau_{r\theta} = 0$$

- Above equation have for ⇒ plane stress (the end faces are free to expand)
  ⇒ plane strain (end faces fixed)

Radial Stress + tangential (circumferential stress) = 4C (constant)

Evaluation of A and C for a thick cylinder

- Consider thick cylinder (fig)
- Inner radius a is subjected to a pressure P_a and outer surface of radius b subjected to a pressure P_b.



Thick cylinder subjected to internal & external pressure.

Boundary condition for radial stress

$$\sigma_{rr}(r=a) = -P_a \Rightarrow \frac{A}{a^2} + 2C = -P_a \quad \text{—— (1)}$$

$$\sigma_{rr}(r=b) = -P_b \Rightarrow \frac{A}{b^2} + 2C = -P_b \quad \text{—— (2)}$$

eq (1) − (2) $\quad A\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = -P_a + P_b$

$$A\left(\frac{b^2-a^2}{a^2b^2}\right) = P_b - P_a$$

$$\boxed{A = \frac{(P_b - P_a)a^2b^2}{b^2-a^2}}$$

eq (1) ⇒ $2C = -P_a - \frac{A}{a^2}$

$$2C = -P_a - \frac{(P_b-P_a)a^2b^2}{(b^2-a^2)a^2}$$

$$2C = -P_a - \frac{(P_b-P_a)a^2b^2}{a^2(b^2-a^2)}$$

$$= \frac{-P_a a^2b^2 + P_a a^4 - P_b a^2b^2 + P_a a^2b^2}{a^2(b^2-a^2)}$$

$$= \frac{a^2(P_a a^2 - P_b b^2)}{a^2(b^2-a^2)}$$

$$\boxed{2C = \frac{P_a a^2 - P_b b^2}{b^2-a^2}}$$

putting value of A and C in eq: of σ_{rr} and $\sigma_{\theta\theta}$ eq (1) and (2)

$$\sigma_{rr}: \frac{A}{r^2} + 2C = \frac{1}{r^2}\left[\frac{(P_b-P_a)a^2b^2}{b^2-a^2}\right] + \frac{P_a a^2 - P_b b^2}{b^2-a^2} \quad \text{(A)}$$

$$\sigma_{\theta\theta} = \frac{-A}{r^2} + 2C = \frac{-1}{r^2}\left[\frac{(P_b-P_a)a^2b^2}{b^2-a^2}\right] + \frac{P_a a^2 - P_b b^2}{b^2-a^2} \quad \text{(B)}$$

Above eq called as lames equation for Radial & tangential Stress

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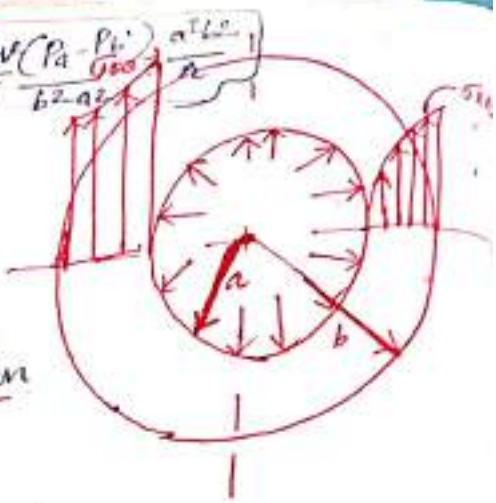
Radial displacement $u_r = \frac{1-\nu}{E}\frac{P_a a^2 - P_b b^2}{b^2-a^2} r + \frac{1+\nu}{E}\frac{(P_a - P_b)}{b^2-a^2}\frac{a^2 b^2}{r}$

(Very Important Figure)



Variation of radial & tangential stress in thick cylinder

$\sigma_r + \sigma_\theta$ = constant throughout thickness

According to Hooke's Law, stresses σ_r and σ_θ produce uniform deformation in z-direction

Axial Stress in plane strain condition

End faces of cylinder are fixed

∴ plane strain condition with $\epsilon_{zz} = 0$

Hooke's Law $\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta}))$

Radial displacement $\epsilon_{zz} = 0 \Rightarrow \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \nu\left(\frac{A}{r^2} + 2C - \frac{A}{r^2} + 2C\right)$

$= 4\nu C$

$$\sigma_{zz} = 2\nu\left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right]$$

$$u_r = \frac{1-\nu}{E}\frac{P_a a^2 - P_b b^2}{b^2-a^2} r + \frac{1+\nu}{E}\frac{(P_a - P_b)}{b^2-a^2}\frac{a^2 b^2}{r}$$

Maximum Shear Stress

$$\tau_{max} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{2} = \frac{A}{r^2} = \frac{(P_a - P_b)a^2 b^2}{r^2(b^2-a^2)}$$

Special cases

① Internal pressure only

Here Internal pressure = P_a
External pressure = $P_b = 0$

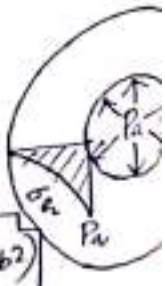
Boundary condition $r = a$ $b = 0$

eq-(B) [illegible] $(P_b = 0,\ r = a)$ [illegible]

$$\sigma_r = \frac{1}{r^2}\left[\frac{P_b a^2 b^2 - P_a a^2 b^2}{b^2-a^2}\right] + \left[\frac{P_a a^2 - P_b b^2}{b^2-a^2}\right] = \frac{1}{r^2}\left[\frac{-P_a a^2 b^2 + P_a a^2}{b^2-a^2}\right]$$

$$\sigma_r = \frac{P_a a^2}{b^2-a^2}\left[1 - \frac{b^2}{r^2}\right] \quad \text{(Compressive Stress)}$$

$$\sigma_\theta = \frac{P_a a^2}{b^2-a^2}\left[1 + \frac{b^2}{r^2}\right] \quad \text{(Tensile Stress)}$$



$$\sigma_{\theta max} = \frac{P_a}{b^2-a^2}(a^2+b^2)$$

Here $P_b = 0$, $r = a$

$$\tau_{max} = \frac{(P_a - 0)a^2 b^2}{a^2(b^2-a^2)} = \frac{+P_a b^2}{b^2-a^2}$$

Radial displacement

$$u_a = \frac{a P_a}{E}\left(\frac{a^2+b^2}{b^2-a^2} + \nu\right)$$

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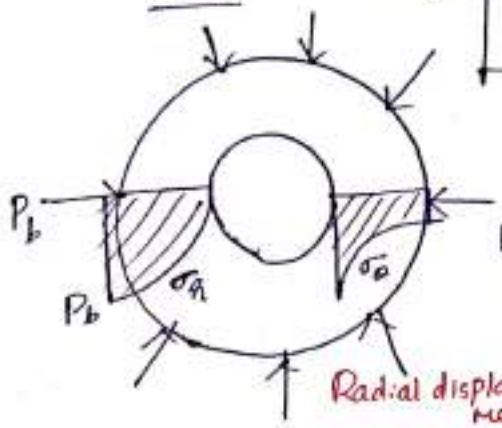
External pressure only

external pressure = P_b

internal pressure $P_a = 0$

$A = b$



Sub: Mean eq: except

$$\sigma_r = \frac{-P_b b^2}{b^2 - a^2}\left[1 - \frac{a^2}{r^2}\right]$$

$$\sigma_\theta = \frac{-P_b b^2}{b^2 - a^2}\left[1 + \frac{a^2}{r^2}\right]$$

} compressive, $\sigma_\theta > \sigma_r$

Max: value inner surface of cylinder

$$\sigma_{\theta max} = \frac{-2b^2 P_b}{b^2 - a^2}$$

If $a = 0$, on inner hole

Stresses are uniformly distributed across thickness $\sigma_\theta = \sigma_r = -P_b$

Radial displacement

$$u_b = \frac{-bP_b}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} - \nu\right)$$

① A thick cylinder of inner radius 8cm and outer radius 12cm is subjected to an external pressure of −0.09 Mpa. Internal pressure is atmospheric

(or)

(a) Radial & tangential stress distribution across the thickness

(b) Axial Component of stress, if the ends of cylinder are fixed $\nu = 0.3$

(c) Max: Shear Stress

Solution

$a = 8cm$, $b = 12cm$, $P_b = -0.09 Mpa = -0.009\ KN/cm^2$, $P_a = 0$

(a) $\sigma_{rr} = \frac{1}{r^2}\left[\frac{(P_b - P_a)a^2b^2}{b^2 - a^2}\right] + \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right]$

$= \frac{1}{r^2}\left[\frac{(-0.009 - 0)8^2 \times 12^2}{12^2 - 8^2}\right] + \frac{(-0.009 \times 12^2)}{12^2 - 8^2}$

$\sigma_r = \frac{1}{r^2} \times -1.0368 + 0.0162\ KN/cm^2$

$\sigma_{\theta\theta} = \frac{1}{r^2}\left[\frac{(P_a - P_b)a^2b^2}{b^2 - a^2}\right] + \frac{P_a a^2 - P_b b^2}{b^2 - a^2} = \frac{1}{r^2}\left[\frac{(0 - (-0.009)8^2 \times 12^2}{12^2 - 8^2}\right]$

$+ \frac{0 - (-0.009 \times 12^2)}{12^2 - 8^2} = \frac{1.0368}{r^2} + 0.0162\ KN/cm^2$

(b) Fixed end face plane strain condition

Valid $\sigma_{zz} = 4\vartheta C = 2\nu\left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] = 2 \times 0.3\left[\frac{0 - (-0.009 \times 12^2)}{12^2 - 8^2}\right]$

$= 97.2 Mpa$

(c) Max: Shear Stress τ_{max} $(r = a)$

$$\tau_{max} = \frac{A}{r^2} = \frac{(P_b - P_a)a^2b^2}{r^2(b^2 - a^2)} = \frac{(-0.009 - 0)8^2 \times 12^2}{8^2(12^2 - 8^2)} = -0.162\ Mpa$$

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$$\sigma_r = \frac{150\times50^2}{200^2-50^2}\left(1-\frac{200^2}{r^2}\right) = -150\text{Mpa} \quad (r=a)$$

$$\Rightarrow \sigma_r = \frac{150\times50^2}{200^2-50^2}\left(1-\frac{200^2}{200^2}\right) = 0 \quad (r=b)$$

$$\Rightarrow \sigma_\theta = \frac{Pa a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{150\times50^2}{200^2-50^2}\left(1+\frac{200^2}{50^2}\right) = 170\text{Mpa} \quad (r=a)$$

$$\sigma_\theta = \frac{150\times50^2}{200^2-50^2}\left(1+\frac{200^2}{200^2}\right) = 80\text{Mpa} \quad (r=b)$$

(6) A thick cylinder of inner radius 10cm & outer radius 15cm subjected to internal pressure of $12N/mm^2$. Determine radial & Hoop Stress of cylinder at the inner and outer surface.

Solution

$a = 10cm$ (Internal radius)

$b = 15cm$ (External radius)

$P_a = 12N/mm^2$ (Internal pressure)

$$\sigma_r = \frac{Pa a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)$$

at $r=a$, $\sigma_r = -Pa = -12N/mm^2$

$r=b$, $\sigma_r = \frac{12\times10^2}{15^2-10^2}\left(1-\frac{15^2}{15^2}\right) = 0$

Hoop Stress or tangential Stress

$$\sigma_\theta = \frac{Pa a^2}{b^2-a^2}\left(1+\frac{b^2}{a^2}\right)$$

$r=a$, $\sigma_\theta = \frac{12\times10^2}{15^2-10^2}\left(1+\frac{15^2}{10^2}\right)$

$= 31.2N/mm^2$

$r=b$, $\sigma_\theta = \frac{12\times10^2}{15^2-10^2}\left(1+\frac{15^2}{15^2}\right)$

$= 19.2N/mm^2$

Syllabus for the Examination

Syllabus: An Objective Type Test (OMR Valuation) based on the qualification prescribed for the post. Main Topics - Part I General Knowledge, Current Affairs & Renaissance in Kerala. Part II : General English. Part III : Simple Arithmetic & Mental Ability. (Maximum Marks : 100). (Duration: 1 hour 15 minutes). (Medium of Questions : English).

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Rotating Discs

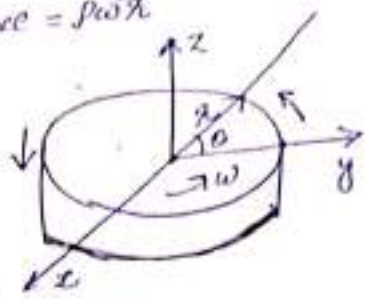
- Stresses induced in a disc due to Rotation

Assumption

① Thickness of disc is small in comparison with radius (σ_r and σ_θ do not varies over thickness)

② Stresses along axis of Rotation ($\sigma_z = 0$) zero

③ Body force in radial direction: Inertia force $= \rho\omega^2 r$



Equilibrium eq:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0$$

$$\Rightarrow r\frac{\partial \sigma_r}{\partial r} + \sigma_r - \sigma_\theta + \rho\omega^2 r^2 = 0$$

$$\boxed{\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0} \quad -①$$

Strain-displacement equation

$$\epsilon_r = \frac{du_r}{dr}, \quad \epsilon_\theta = \frac{u_r}{r}$$

$$\epsilon_r = \frac{d}{dr}(r\epsilon_\theta) \quad -Ⓐ$$

$$\boxed{G_r = \epsilon_\theta + r\frac{d\epsilon_\theta}{dr}} \quad -Ⓑ$$

Hooks law ($\sigma_z = 0$)

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \quad -②$$

Sub: eq ② in Ⓐ

$$\epsilon_r = \frac{1}{E}\frac{d}{dr}(r\sigma_\theta - \nu r\sigma_r)$$

Differential eq:

$$\boxed{r^2\frac{d^2\psi}{dr^2} + r\frac{d\psi}{dr} - \psi + (3+\vartheta)\rho\omega^2 r^3 = 0}$$

Solution of differential eq:

$$\boxed{\begin{aligned} \psi &= Cr + \frac{D}{r} - \frac{(3+\vartheta)}{8}\rho\omega^2 r^3 \\ \sigma_r &= C + \frac{D}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \\ \sigma_\theta &= C - \frac{D}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2 \end{aligned}}$$

Constant C and D Find using Boundary condition

Solid disc

- In solid disc radius $= b$

no external force on outer surface ($\sigma_r = 0, r = b, D = 0$ (when $r = 0$))

$D = 0$ (when $r = 0$) $\Rightarrow$ Sub: $\sigma_r = 0, r = b, D = 0$ in eq: of σ_r

$$\therefore \quad 0 = C - \frac{(3+\vartheta)}{8}\rho\omega^2 r^2 \Rightarrow \boxed{C = \frac{3+\vartheta}{8}\rho\omega^2 r^2} \Rightarrow \boxed{C = \frac{3+\vartheta}{8}\rho\omega^2 b^2}$$

$$\therefore \sigma_r = \frac{3+\vartheta}{8}\rho\omega^2 b^2 - \frac{3+\vartheta}{8}\rho\omega^2 r^2$$

$$\boxed{\sigma_r = \frac{(3+\vartheta)}{8}\rho\omega^2(b^2 - r^2)}$$

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$$\sigma_\theta = \frac{C-D}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$

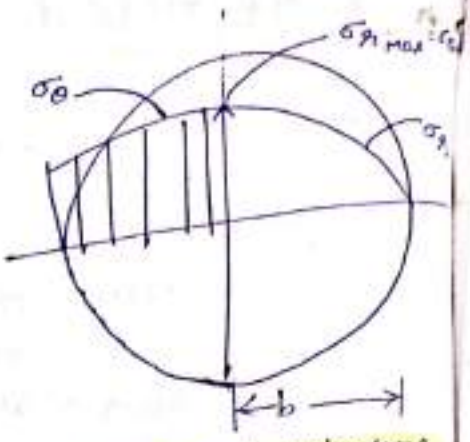
$$= \left(\frac{3+\nu}{8}\right)\rho\omega^2 b^2 - 0 - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$

$$\boxed{\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)b^2 - (1+3\nu)r^2\right]}$$

at $r=0$ $\quad \boxed{\sigma_{r\,max} = \sigma_{\theta\,max} = \frac{(3+\nu)}{8}\rho\omega^2 b^2}$

at $r=b$ $\quad \sigma_\theta = \frac{1-\nu}{4}\rho\omega^2 b^2$ & $\sigma_r = 0$

Max: Stress at Centre of disc



Stress distribution

Disc with Central Hole

- Disc with Internal radius = a
  External radius = b
- Inner & outer surface are not subjected to external loading

Boundary condition $\Rightarrow r=a, \sigma_r = 0$

$r=b, \sigma_r = 0$

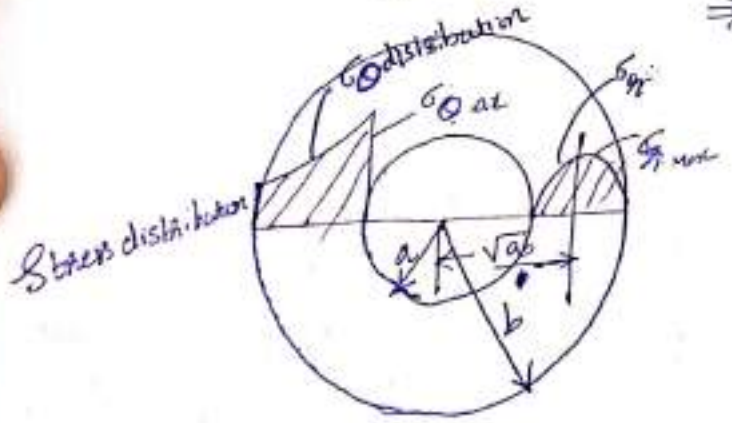
$$\sigma_r = \frac{C+D}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \Rightarrow 0 = \frac{C+D}{a^2} - \frac{(3+\nu)}{8}\rho\omega^2 a^2 \quad (-)$$

$$\Rightarrow 0 = \frac{C+D}{b^2} - \frac{(3+\nu)}{8}\rho\omega^2 b^2$$

$$0 = D\left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{3+\nu}{8}\rho\omega^2(a^2-b^2)$$

$$0 = D\left(\frac{b^2-a^2}{a^2b^2}\right) - \frac{3+\nu}{8}\rho\omega^2(a^2-b^2)$$

$$D\frac{(a^2-b^2)}{a^2b^2} = -\frac{3+\nu}{8}\rho\omega^2(a^2-b^2)$$



$$\boxed{D = -\frac{(3+\nu)}{8}\rho\omega^2 a^2 b^2}$$

Sub above eq

$$\boxed{C = \frac{3+\nu}{8}\rho\omega^2(b^2+a^2)}$$

$$\boxed{\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2+a^2-\frac{a^2b^2}{r^2}-r^2\right)}$$

$$\boxed{\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2+a^2+\frac{a^2b^2}{r^2}-\frac{1+3\nu}{3+\nu}r^2\right)}$$

when $r=a$

$$\boxed{\sigma_{\theta max} = \left(\frac{3+\nu}{8}\right)\rho\omega^2 b^2\left(1+\frac{(1-\nu)}{3+\nu}\cdot\frac{a^2}{b^2}\right)}$$

when $r=\sqrt{ab}$

$$\boxed{\sigma_{r\,max} = \frac{3+\nu}{8}\rho\omega^2(b-a)^2}$$

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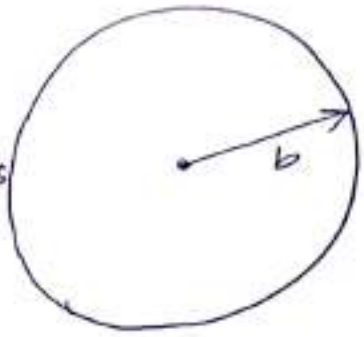
① A solid steel shaft of diameter 15cm running at 5000 rpm

Take $E = 214 \times 10^6$ kPa, $\rho = 8050$ kg/m3 and $\vartheta = 0.3$

Determine (a) Maximum values of radial stress and tangential stress
(b) outer radius of disc under running conditions

Solution



Given data → $b = 7.5$cm

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 5000}{60} = 523.6 \text{ rad/s}$$

(a) Max: radial & tangential stress occur at center ($r = 0$)

$$(\sigma_r)_{max} = (\sigma_\theta)_{max} = \frac{(3+\vartheta)}{8}\rho\omega^2 b^2 = \frac{(3+0.3)}{8} 8050 (5236)^2 \times (7.5\times10^{-2})^2$$

$$= 1.2013 \times 10^6 \text{ pa}$$

(b) Tangential stress at rim

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\vartheta)b^2 - (1+3\vartheta)r^2\right]$$

when $r = b = 7.5$cm

$$\sigma_\theta = \frac{8050 \times 523 b^2}{8}\left[(3+0.3)(7.5\times10^{-2})^2 - (1+3\times0.3)(7.5\times10^{-2})^2\right]$$

$$= 2.1725 \times 10^6 \text{ pa}$$

Displacement of a point on disc $\Rightarrow u_r = r\epsilon_\theta = \frac{r}{E}(\sigma_\theta - \vartheta\sigma_r)$

$$u_r(r=b) = \frac{b}{E}(\sigma_\theta - \vartheta\sigma_r) = \frac{7.5\times10^{-2}}{214\times10^9}(2.1725\times10^6 - 0)$$

$$= 7.6138 \times 10^{-7} \text{ m}$$

Radius of disc under running $= b + u_r = 7.5 + 7.6138 \times 10^{-5}$

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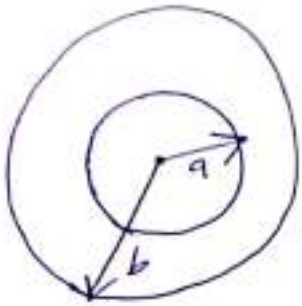
② A disc with a hole has inner diameter of 6cm and outer diameter of ...
If disc is running at 8000rpm
Determine (a) The distribution of radial & tangential stress
(b) Find Max: Value of radial & tangential stress
Given $\nu = 0.3$, $\rho = 8050 kg/m^3$



Solution

$a = 3, b = 15$ (cm)

$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 8000}{60} = 100\pi$ rad/s, $\nu = 0.3$

(a) $\sigma_r(r) = \frac{3+\nu}{8}\rho\omega^2\left(b^2+a^2-\frac{a^2b^2}{r^2}-r^2\right)$

$= \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2-\frac{3^2 15^2}{r^2}-r^2\right) \times 10^{-4}$

when $\sigma_r(r=a) = \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2-\frac{3^2\times 15^2}{3^2}-3^2\right)\times 10^{-4} =$ —

$\sigma_r(r=b) = \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2-\frac{3^2\times 15^2}{15^2}-15^2\right)\times 10^{-4} =$ —

$\sigma_\theta(r) = \frac{3+\nu}{8}\rho\omega^2\left(b^2+a^2+\frac{a^2b^2}{r^2}-\frac{1+3\nu}{3+\nu}r^2\right)$

$= \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2+\frac{3^2\times 15^2}{r^2}-\frac{(1+3\times 0.3)}{3+0.3}r^2\right)$

when $\sigma_\theta(r=a) = \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2+\frac{3^2\times 15^2}{3^2}-\frac{(1+3\times 0.3)}{3+0.3}3^2\right)$

$=$ _____

when $\sigma_\theta(r=b) = \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2+\frac{3^2\times 15^2}{15^2}-\frac{(1+3\times 0.3)}{3+0.3}15^2\right)$

$=$ _____

(b) Max: value of Radial Stress (σ_r) occurs at

$r = \sqrt{ab} = \sqrt{3\times 15} = 6.7082$ cm

$(\sigma_r)_{max} = \frac{3+\nu}{8}\rho\omega^2 8050 \times (100\pi)^2 \left(15^2+3^2-\frac{3^2 15^2}{6.7082^2}\right) =$ 47...

Max: tangential stress occurs at $r = a = 3$cm

$\sigma_\theta(max) = \frac{3+\nu}{8}\rho\omega^2\left(b^2+a^2+\frac{a^2b^2}{r^2}-\frac{1+3\nu}{3+\nu}r^2\right)$

$= \frac{3+0.3}{8} \times 8050 \times (100\pi)^2 \left(15^2+3^2+\frac{3\times 15^2}{3^2}-\frac{(1+3\times 0.3)}{3+0.3}3^2\right)$

$= 1.4861 \times 10^4$ kPa

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